

# Kuranishi homology and Kuranishi cohomology: a User's Guide

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## Abstract

A *Kuranishi space* is a topological space equipped with a *Kuranishi structure*, defined by Fukaya and Ono. Kuranishi structures occur naturally on many moduli spaces in differential geometry, and in particular, in moduli spaces of stable  $J$ -holomorphic curves in Symplectic Geometry.

This paper is a summary of the author's book [12]. Let  $Y$  be an orbifold, and  $R$  a  $\mathbb{Q}$ -algebra. We shall define a new homology theory of  $Y$ , *Kuranishi homology*  $KH_*(Y; R)$ , using a chain complex  $KC_*(Y; R)$  spanned by isomorphism classes  $[X, \mathbf{f}, \mathbf{G}]$ , where  $X$  is a compact, oriented Kuranishi space with corners,  $\mathbf{f} : X \rightarrow Y$  is strongly smooth, and  $\mathbf{G}$  is some extra *gauge-fixing data* for  $(X, \mathbf{f})$ . The purpose of  $\mathbf{G}$  is to ensure the automorphism groups  $\text{Aut}(X, \mathbf{f}, \mathbf{G})$  are finite, which is necessary to get a well-behaved homology theory. Our main result is that  $KH_*(Y; R)$  is isomorphic to singular homology.

We define Poincaré dual *Kuranishi (co)homology*  $KH^*(Y; R)$ , which is isomorphic to compactly-supported cohomology, using a cochain complex  $KC^*(Y; R)$  spanned by  $[X, \mathbf{f}, \mathbf{C}]$ , where  $X$  is a compact Kuranishi space with corners,  $\mathbf{f} : X \rightarrow Y$  is a cooriented strong submersion, and  $\mathbf{C}$  is *co-gauge-fixing data*. We also define simpler theories of *Kuranishi bordism* and *Kuranishi cobordism*  $KB_*, KB^*(Y; R)$ , for  $R$  a commutative ring.

These theories are powerful new tools in Symplectic Geometry. Moduli spaces of  $J$ -holomorphic curves define (co)chains directly in Kuranishi (co)homology or Kuranishi (co)bordism. This hugely simplifies the formation of virtual cycles, as there is no longer any need to perturb moduli spaces. The theory has applications to Lagrangian Floer cohomology, String Topology, and the Gopakumar–Vafa Integrality Conjecture.

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# 1 Introduction

A *Kuranishi space* is a topological space with a *Kuranishi structure*, defined by Fukaya and Ono [10, 11]. Let  $Y$  be an orbifold and  $R$  a  $\mathbb{Q}$ -algebra. In the book [12] the author develops *Kuranishi homology*  $KH_*(Y; R)$  and *Kuranishi cohomology*  $KH^*(Y; R)$ . The (co)chains in these theories are of the form  $[X, \mathbf{f}, \mathbf{G}]$  where  $X$  is a compact Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is a strongly smooth map, and  $\mathbf{G}$  is some extra *gauge-fixing data*. We prove  $KH_*(Y; R)$  is isomorphic to singular homology  $H_*^{\text{si}}(Y; R)$ , and  $KH^*(Y; R)$  is isomorphic to compactly-supported cohomology  $H_{\text{cs}}^*(Y; R)$ . We also define *Kuranishi bordism*  $KB_*(Y; R)$  and *Kuranishi cobordism*  $KB^*(Y; R)$ , for  $R$  a commutative ring.

This paper is a brief introduction to selected parts of [12]. The length of [12] (presently 290 pages) is likely to deter people from reading it, but the main ideas can be summarized much more briefly, and that is what this paper tries to do. A User's Guide, say for a car or a computer, should give you a broad overview of how the machine actually works, and instructions on how to use it in practice, but it should probably not tell you exactly where the carburettor is, or how the motherboard is wired. This paper is written in the same spirit. It should provide you with sufficient background to understand the sequels [2, 13, 14] (once I get round to writing them), and also to decide whether Kuranishi (co)homology is a good tool to use in problems you are interested in.

Here are the main areas of [12] that we will *not* cover. In [12] we also define *effective Kuranishi (co)homology*  $KH_*^{\text{ef}}, KH_{\text{ec}}^*(Y; R)$ , a variant on Kuranishi homology that has the advantage that it works for all commutative rings  $R$ , including  $R = \mathbb{Z}$ , and is isomorphic to  $H_*^{\text{si}}, H_{\text{cs}}^*(Y; R)$ , but has the disadvantage

of restrictions on the Kuranishi spaces  $X$  allowed in chains  $[X, \mathbf{f}, \mathbf{G}]$ , which limits its applications in Symplectic Geometry. Similarly, [12, Ch. 5] actually defines five different kinds of Kuranishi (co)bordism, but below we consider only the simplest. The applications to Symplectic Geometry in [12, Ch. 6] are omitted.

Kuranishi (co)homology is intended primarily as a tool for use in areas of Symplectic Geometry involving  $J$ -holomorphic curves, and will be applied in the sequels [2, 13–15]. Kuranishi structures occur naturally on many moduli spaces in Differential Geometry. For example, if  $(M, \omega)$  is a compact symplectic manifold with almost complex structure  $J$  then the moduli space  $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$  of stable  $J$ -holomorphic curves of genus  $g$  with  $m$  marked points in class  $\beta$  in  $H_2(M; \mathbb{Z})$  is a compact Kuranishi space with a strongly smooth map  $\prod_i \mathbf{ev}_i : \bar{\mathcal{M}}_{g,m}(M, J, \beta) \rightarrow M^m$ . By choosing some gauge-fixing data  $\mathbf{G}$  we define a cycle  $[\bar{\mathcal{M}}_{g,m}(M, J, \beta), \prod_i \mathbf{ev}_i, \mathbf{G}]$  in  $KC_*(M^m; \mathbb{Q})$  whose homology class  $[[\bar{\mathcal{M}}_{g,m}(M, J, \beta), \prod_i \mathbf{ev}_i, \mathbf{G}]]$  in  $KH_*(M^m; \mathbb{Q}) \cong H_*^{\text{si}}(M^m; \mathbb{Q})$  is a *Gromov–Witten invariant* of  $(M, \omega)$ , and is independent of the choice of almost complex structure  $J$ .

In the conventional definitions of symplectic Gromov–Witten invariants [11, 16, 17, 19], one must define a *virtual cycle* for  $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$ . This is a complicated process, involving many arbitrary choices: first one must perturb the moduli space, morally over  $\mathbb{Q}$  rather than  $\mathbb{Z}$ , so that it becomes something like a manifold. Then one must triangulate the perturbed moduli space by simplices to define a cycle in the singular chains  $C_*^{\text{si}}(M^m; \mathbb{Q})$ . The Gromov–Witten invariant is the homology class of this virtual cycle. By using Kuranishi (co)homology as a substitute for singular homology, this process of defining virtual cycles becomes much simpler and less arbitrary. *The moduli space is its own virtual cycle*, and we eliminate the need to perturb moduli spaces and triangulate by simplices.

The real benefits of the Kuranishi (co)homology approach come not in closed Gromov–Witten theory, where the moduli spaces are Kuranishi spaces without boundary, but in areas such as open Gromov–Witten theory, Lagrangian Floer cohomology [10], Contact Homology [8], and Symplectic Field Theory [9], where the moduli spaces are Kuranishi spaces *with boundary and corners*, and their boundaries are identified with fibre products of other moduli spaces.

In the conventional approach, one must choose virtual chains for each moduli space, which must be compatible at the boundary with intersection products of choices of virtual chains for other moduli spaces. This business of boundary compatibility of virtual chains is horribly complicated and messy, and a large part of the 1385 pages of Fukaya, Oh, Ohta and Ono’s work on Lagrangian Floer cohomology [10] is devoted to dealing with it. Using Kuranishi cohomology, because we do not perturb moduli spaces, choosing virtual chains with boundary compatibility is very easy, and Lagrangian Floer cohomology can be reformulated in a much more economical way, as we will show in [12, §6.6] and [2].

An important feature of these theories is that *Kuranishi homology and cohomology are very well behaved at the (co)chain level*, much better than singular homology, say. For example, Kuranishi cochains  $KC^*(Y; R)$  have a supercommutative, associative cup product  $\cup$ , cap products also work well at

the (co)chain level, and there is a well-behaved functor from singular chains  $C_*^{\text{si}}(Y; R)$  to Kuranishi chains  $KC_*(Y; R)$ . Because of this, the theories may also have applications in other areas which may not be directly related to Kuranishi spaces, but which need a (co)homology theory of manifolds or orbifolds with good (co)chain-level behaviour. In [14] we will apply Kuranishi (co)chains to reformulate the String Topology of Chas and Sullivan [5], which involves chains on infinite-dimensional loop spaces. Another possible area is Costello's approach to Topological Conformal Field Theories [7], which involves a choice of chain complex for homology, applied to moduli spaces of Riemann surfaces.

As well as Kuranishi homology and cohomology, in [12, Ch. 5] we also define *Kuranishi bordism*  $KB_*(Y; R)$  and *Kuranishi cobordism*  $KB^*(Y; R)$ . These are simpler than Kuranishi (co)homology, being spanned by  $[X, \mathbf{f}]$  for  $X$  a compact Kuranishi space *without boundary* and  $\mathbf{f} : X \rightarrow Y$  strongly smooth, and do not involve gauge-fixing data. In contrast to Kuranishi (co)homology which is isomorphic to conventional homology and compactly-supported cohomology, these are new topological invariants, and we show that they are very large — for instance, if  $Y \neq \emptyset$  and  $R \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$  then  $KB_{2k}(Y; R)$  is infinitely generated over  $R$  for all  $k \in \mathbb{Z}$ .

In [12, §6.2] we define new Gromov–Witten type invariants  $[\overline{\mathcal{M}}_{g,m}(M, J, \beta), \prod_i \mathbf{ev}_i]$  in Kuranishi bordism  $KB_*(M^m; \mathbb{Z})$ . Since these are defined in groups over  $\mathbb{Z}$ , not  $\mathbb{Q}$ , the author expects that Kuranishi (co)bordism will be useful in studying *integrality properties* of Gromov–Witten invariants. In [12, §6.3] we outline an approach to proving the Gopakumar–Vafa Integrality Conjecture for Gromov–Witten invariants of Calabi–Yau 3-folds, which the author hopes to take further in [15].

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## 2 Kuranishi spaces

*Kuranishi spaces* were introduced by Fukaya and Ono [10, 11], and are important in Symplectic Geometry because moduli spaces of stable  $J$ -holomorphic curves in symplectic manifolds are Kuranishi spaces. We use the definitions and notation of [12, §2], which have some modifications from those of Fukaya–Ono.

### 2.1 Manifolds and orbifolds with corners and g-corners

In [12] we work with four classes of manifolds, in increasing order of generality: *manifolds without boundary*, *manifolds with boundary*, *manifolds with corners*, and *manifolds with generalized corners* or *g-corners*. The first three classes are fairly standard, although the author has not found a reference for foundational material on manifolds with corners. Manifolds with g-corners are new, as far as

the author knows. The precise definitions of these classes of manifolds are given in [12, §2.1]. Here are the basic ideas:

- An *n-dimensional manifold without boundary* is locally modelled on open sets in  $\mathbb{R}^n$ .
- An *n-dimensional manifold with boundary* is locally modelled on open sets in  $\mathbb{R}^n$  or  $[0, \infty) \times \mathbb{R}^{n-1}$ .
- An *n-dimensional manifold with corners* is locally modelled on open sets in  $[0, \infty)^k \times \mathbb{R}^{n-k}$  for  $k = 0, \dots, n$ .
- A *polyhedral cone*  $C$  in  $\mathbb{R}^n$  is a subset of the form

$$C = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_1^i x_1 + \dots + a_n^i x_n \geq 0, \ i = 1, \dots, k\},$$

where  $a_j^i \in \mathbb{R}$  for  $i = 1, \dots, k$  and  $j = 1, \dots, n$ .

An *n-dimensional manifold with g-corners* is roughly speaking locally modelled on open sets in polyhedral cones  $C$  in  $\mathbb{R}^n$  with nonempty interior  $C^\circ$ . Since  $[0, \infty)^k \times \mathbb{R}^{n-k}$  is a polyhedral cone  $C$  with  $C^\circ \neq \emptyset$ , manifolds with corners are examples of manifolds with g-corners. In fact the subsets in  $\mathbb{R}^n$  used as local models for manifolds with g-corners are more general than polyhedral cones, but this extra generality is only needed for technical reasons in the proof of Theorem 3.3.

Here are some examples. The line  $\mathbb{R}$  is a 1-manifold without boundary; the interval  $[0, 1]$  is a 1-manifold with boundary; the square  $[0, 1]^2$  is a 2-manifold with corners; and the octahedron

$$O = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$$

in  $\mathbb{R}^3$  is a 3-manifold with g-corners. It is not a manifold with corners, since four 2-dimensional faces of  $O$  meet at the vertex  $(1, 0, 0)$ , but three 2-dimensional faces of  $[0, \infty)^3$  meet at the vertex  $(0, 0, 0)$ , so  $O$  near  $(1, 0, 0)$  is not locally modelled on  $[0, \infty)^3$  near  $(0, 0, 0)$ .

Manifolds  $X$  with boundary, corners, or g-corners have a well-behaved notion of *boundary*  $\partial X$ . To motivate the definition, consider  $[0, \infty)^2$  in  $\mathbb{R}^2$ . If we took  $\partial([0, \infty)^2)$  to be the subset  $([0, \infty) \times \{0\}) \cup (\{0\} \times [0, \infty))$  of  $[0, \infty)^2$ , then  $\partial([0, \infty)^2)$  would not be a manifold with corners near  $(0, 0)$ . Instead, we take  $\partial([0, \infty)^2)$  to be the *disjoint union* of the two boundary strata  $[0, \infty) \times \{0\}$  and  $\{0\} \times [0, \infty)$ . This is a manifold with boundary, but now  $\partial([0, \infty)^2)$  is *not a subset of*  $[0, \infty)^2$ , since two points in  $\partial([0, \infty)^2)$  correspond to  $(0, 0)$  in  $[0, \infty)^2$ .

We define the *boundary*  $\partial X$  of an  $n$ -manifold  $X$  with (g-)corners to be the set of pairs  $(p, B)$ , where  $p \in X$  and  $B$  is a local choice of connected  $(n-1)$ -dimensional boundary stratum of  $X$  containing  $p$ . Thus, if  $p$  lies in a codimension  $k$  corner of  $X$  locally modelled on  $[0, \infty)^k \times \mathbb{R}^{n-k}$  then  $p$  is represented by  $k$  distinct points  $(p, B_i)$  in  $\partial X$  for  $i = 1, \dots, k$ . Then  $\partial X$  is an  $(n-1)$ -manifold with (g-)corners. Note that  $\partial X$  is not a subset of  $X$ , but has a natural immersion  $\iota : \partial X \rightarrow X$  mapping  $(p, B) \mapsto p$ . Often we suppress  $\iota$ , and talk of restricting data on  $X$  to  $\partial X$ , when really we mean the pullback by  $\iota$ .

If  $X$  is a  $n$ -manifold with (g-)corners then  $\partial^2 X$  is an  $(n-2)$ -manifold with (g-)corners. Points of  $\partial^2 X$  may be written  $(p, B_1, B_2)$ , where  $p \in X$  and  $B_1, B_2$  are distinct local boundary components of  $X$  containing  $p$ . There is a natural, free involution  $\sigma : \partial^2 X \rightarrow \partial^2 X$  mapping  $\sigma : (p, B_1, B_2) \mapsto (p, B_2, B_1)$ , which is orientation-reversing if  $X$  is oriented. This involution is important in questions to do with extending data defined on  $\partial X$  to  $X$ . For example, if  $f : \partial X \rightarrow \mathbb{R}$  is a smooth function, then a necessary condition for there to exist smooth  $g : X \rightarrow \mathbb{R}$  with  $g|_{\partial X} \equiv f$  is that  $f|_{\partial^2 X}$  is  $\sigma$ -invariant, and if  $X$  has corners (not g-corners) then this condition is also sufficient.

*Orbifolds* are a generalization of manifold, which allow quotients by finite groups. Again, we define orbifolds *without boundary*, or *with boundary*, or *with corners*, or *with g-corners*, where orbifolds without boundary are locally modelled on quotients  $\mathbb{R}^n/\Gamma$  for  $\Gamma$  a finite group acting linearly on  $\mathbb{R}^n$ , and similarly for the other classes. (We do not require  $\Gamma$  to act *effectively*, so we cannot regard  $\Gamma$  as a subgroup of  $\mathrm{GL}(n, \mathbb{R})$ .) Orbifolds (then called *V-manifolds*) were introduced by Satake [18], and a book on orbifolds is Adem et al. [1]. Note however that the right definition of smooth maps of orbifolds is not that given by Satake, but the more complex notion of *morphisms of orbifolds* in [1, §2.4]. When we do not specify otherwise, by a manifold or orbifold, we always mean a manifold or orbifold with g-corners, the most general class.

Let  $X, Y$  be manifolds of dimensions  $m, n$ . *Smooth maps*  $f : X \rightarrow Y$  are continuous maps which are locally modelled on smooth maps from  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ . A smooth map  $f$  induces a morphism of vector bundles  $df : TX \rightarrow f^*(TY)$  on  $X$ , where  $TX, TY$  are the tangent bundles of  $X$  and  $Y$ . For manifolds  $X, Y$  without boundary, we call a smooth map  $f : X \rightarrow Y$  a *submersion* if  $df : TX \rightarrow f^*(TY)$  is a surjective morphism of vector bundles. If  $X, Y$  have boundary or (g-)corners, the definition of submersions  $f : X \rightarrow Y$  in [12, §2.1] is more complicated, involving conditions over  $\partial^k X$  and  $\partial^l Y$  for all  $k, l \geq 0$ . When  $\partial Y = \emptyset$ ,  $f$  is a submersion if  $d(f|_{\partial^k X}) : T(\partial^k X) \rightarrow f|_{\partial^k X}^*(TY)$  is surjective for all  $k \geq 0$ .

Let  $X, X', Y$  be manifolds (in any of the four classes above) and  $f : X \rightarrow Y$ ,  $f' : X' \rightarrow Y$  be smooth maps, at least one of which is a submersion. Then the *fibre product*  $X \times_{f,Y,f'} X'$  or  $X \times_Y X'$  is

$$X \times_{f,Y,f'} X' = \{(p, p') \in X \times X' : f(p) = f'(p')\}. \quad (1)$$

It turns out [12, Prop. 2.6] that  $X \times_Y X'$  is a submanifold of  $X \times X'$ , and so is a manifold (in the same class as  $X, X', Y$ ). When  $X, X', Y$  have boundary or (g-)corners, the complicated definition of submersion is necessary to make  $X \times_Y X'$  a submanifold over  $\partial^k X, \partial^{k'} X', \partial^l Y$ .

Fibre products can be defined for orbifolds [12, §2.2], but there are some subtleties to do with stabilizer groups. To explain this, first consider the case in which  $U, U', V$  are manifolds, and  $\Gamma, \Gamma', \Delta$  are finite groups acting on  $U, U', V$  by diffeomorphisms so that  $U/\Gamma, U'/\Gamma', V/\Delta$  are orbifolds, and  $\rho : \Gamma \rightarrow \Delta$ ,  $\rho' : \Gamma' \rightarrow \Delta$  are group homomorphisms, and  $f : U \rightarrow V$ ,  $f' : U' \rightarrow V$  are smooth  $\rho$ - and  $\rho'$ -equivariant maps, at least one of which is a submersion. Then

$f, f'$  induce smooth maps of orbifolds  $f_* : U/\Gamma \rightarrow V/\Delta$ ,  $f'_* : U'/\Gamma' \rightarrow V/\Delta$ , at least one of which is a submersion.

It turns out that the right answer for the orbifold fibre product is

$$(U/\Gamma) \times_{f_*, V/\Delta, f'_*} (U'/\Gamma') = ((U \times U') \times_{f \times f', V \times V, \pi} (V \times \Delta)) / (\Gamma \times \Gamma'). \quad (2)$$

Here  $\pi : V \times \Delta \rightarrow V \times V$  is given by  $\pi : (v, \delta) \mapsto (v, \delta \cdot v)$ , and  $(U \times U') \times_{f \times f', V \times V, \pi} (V \times \Delta)$  is the fibre product of smooth manifolds, and  $\Gamma \times \Gamma'$  acts on the manifold  $(U \times U') \times_{V \times V} (V \times \Delta)$  by diffeomorphism  $(\gamma, \gamma') : ((u, u'), (v, \delta)) \mapsto ((\gamma \cdot u, \gamma' \cdot u'), (\rho(\gamma) \cdot v, \rho'(\gamma') \delta \rho'(\gamma')^{-1}))$ , so that the quotient is an orbifold. Now (2) coincides with (1) for  $X = U/\Gamma$ ,  $X' = U'/\Gamma'$ ,  $Y = V/\Delta$  only if one of  $\rho : \Gamma \rightarrow \Delta$ ,  $\rho' : \Gamma' \rightarrow \Delta$  are surjective; otherwise the projection from (2) to (1) is a finite surjective map, but not necessarily injective.

This motivates the definition of fibre products of orbifolds. Let  $X, X', Y$  be orbifolds, and  $f : X \rightarrow Y$ ,  $f' : X' \rightarrow Y$  be smooth maps, at least one of which is a submersion. Then for  $p \in X$  and  $p' \in X'$  with  $f(p) = q = f'(p')$  in  $Y$  we have morphisms of stabilizer groups  $f_* : \text{Stab}(p) \rightarrow \text{Stab}(q)$ ,  $f'_* : \text{Stab}(p') \rightarrow \text{Stab}(q)$ . Thus we can form the double coset space

$$\begin{aligned} & f_*(\text{Stab}(p)) \backslash \text{Stab}(q) / f'_*(\text{Stab}(p')) \\ &= \{ \{ f_*(\gamma) \delta f'_*(\gamma') : \gamma \in \text{Stab}(p), \gamma' \in \text{Stab}(p') \} : \delta \in \text{Stab}(q) \}. \end{aligned}$$

As a set, we define

$$\begin{aligned} X \times_{f, Y, f'} X' &= \{ (p, p', \Delta) : p \in X, p' \in X', f(p) = f'(p'), \\ & \Delta \in f_*(\text{Stab}(p)) \backslash \text{Stab}(f(p)) / f'_*(\text{Stab}(p')) \}. \end{aligned} \quad (3)$$

We give this an orbifold structure in a natural way, such that if  $(U, \Gamma, \phi)$ ,  $(U', \Gamma', \phi')$ ,  $(V, \Delta, \psi)$  are orbifold charts on  $X, X', Y$  with  $f(\text{Im } \phi), f'(\text{Im } \phi') \subseteq \text{Im } \psi$  then we use (2) to define an orbifold chart on  $X \times_Y X'$ .

## 2.2 Kuranishi structures on topological spaces

Let  $X$  be a paracompact Hausdorff topological space throughout.

**Definition 2.1.** A *Kuranishi neighbourhood*  $(V_p, E_p, s_p, \psi_p)$  of  $p \in X$  satisfies:

- (i)  $V_p$  is an orbifold, which may or may not have boundary or (g-)corners;
- (ii)  $E_p \rightarrow V_p$  is an orbifold vector bundle over  $V_p$ , the *obstruction bundle*;
- (iii)  $s_p : V_p \rightarrow E_p$  is a smooth section, the *Kuranishi map*; and
- (iv)  $\psi_p$  is a homeomorphism from  $s_p^{-1}(0)$  to an open neighbourhood of  $p$  in  $X$ , where  $s_p^{-1}(0)$  is the subset of  $V_p$  where the section  $s_p$  is zero.

**Definition 2.2.** Let  $(V_p, E_p, s_p, \psi_p), (\tilde{V}_p, \tilde{E}_p, \tilde{s}_p, \tilde{\psi}_p)$  be two Kuranishi neighbourhoods of  $p \in X$ . We call  $(\alpha, \hat{\alpha}) : (V_p, \dots, \psi_p) \rightarrow (\tilde{V}_p, \dots, \tilde{\psi}_p)$  an *isomorphism* if  $\alpha : V_p \rightarrow \tilde{V}_p$  is a diffeomorphism and  $\hat{\alpha} : E_p \rightarrow \alpha^*(\tilde{E}_p)$  an isomorphism of orbibundles, such that  $\tilde{s}_p \circ \alpha \equiv \hat{\alpha} \circ s_p$  and  $\tilde{\psi}_p \circ \alpha \equiv \psi_p$ .

We call  $(V_p, \dots, \psi_p), (\tilde{V}_p, \dots, \tilde{\psi}_p)$  *equivalent* if there exist open neighbourhoods  $U_p \subseteq V_p$ ,  $\tilde{U}_p \subseteq \tilde{V}_p$  of  $\psi_p^{-1}(p), \tilde{\psi}_p^{-1}(p)$  such that  $(U_p, E_p|_{U_p}, s_p|_{U_p}, \psi_p|_{U_p})$  and  $(\tilde{U}_p, \tilde{E}_p|_{\tilde{U}_p}, \tilde{s}_p|_{\tilde{U}_p}, \tilde{\psi}_p|_{\tilde{U}_p})$  are isomorphic.

**Definition 2.3.** Let  $(V_p, E_p, s_p, \psi_p)$  and  $(V_q, E_q, s_q, \psi_q)$  be Kuranishi neighbourhoods of  $p \in X$  and  $q \in \psi_p(s_p^{-1}(0))$  respectively. We call a pair  $(\phi_{pq}, \hat{\phi}_{pq})$  a *coordinate change* from  $(V_q, \dots, \psi_q)$  to  $(V_p, \dots, \psi_p)$  if:

- (a)  $\phi_{pq} : V_q \rightarrow V_p$  is a smooth embedding of orbifolds;
- (b)  $\hat{\phi}_{pq} : E_q \rightarrow \phi_{pq}^*(E_p)$  is an embedding of orbibundles over  $V_q$ ;
- (c)  $\hat{\phi}_{pq} \circ s_q \equiv s_p \circ \phi_{pq}$ ;
- (d)  $\psi_q \equiv \psi_p \circ \phi_{pq}$ ; and
- (e) Choose an open neighbourhood  $W_{pq}$  of  $\phi_{pq}(V_q)$  in  $V_p$ , and an orbifold vector subbundle  $F_{pq}$  of  $E_p|_{W_{pq}}$  with  $\phi_{pq}^*(F_{pq}) = \hat{\phi}_{pq}(E_q)$ , as orbifold vector subbundles of  $\phi_{pq}^*(E_p)$  over  $V_q$ . Write  $\hat{s}_p : W_{pq} \rightarrow E_p/F_{pq}$  for the projection of  $s_p|_{W_{pq}}$  to the quotient bundle  $E_p/F_{pq}$ . Now  $s_p|_{\phi_{pq}(V_q)}$  lies in  $F_{pq}$  by (c), so  $\hat{s}_p|_{\phi_{pq}(V_q)} \equiv 0$ . Thus there is a well-defined derivative

$$d\hat{s}_p : N_{\phi_{pq}(V_q)}V_p \rightarrow (E_p/F_{pq})|_{\phi_{pq}(V_q)},$$

where  $N_{\phi_{pq}(V_q)}V_p$  is the normal orbifold vector bundle of  $\phi_{pq}(V_q)$  in  $V_p$ . Pulling back to  $V_q$  using  $\phi_{pq}$ , and noting that  $\phi_{pq}^*(F_{pq}) = \hat{\phi}_{pq}(E_q)$ , gives a morphism of orbifold vector bundles over  $V_q$ :

$$d\hat{s}_p : \frac{\phi_{pq}^*(TV_p)}{(d\phi_{pq})(TV_q)} \longrightarrow \frac{\phi_{pq}^*(E_p)}{\hat{\phi}_{pq}(E_q)}. \quad (4)$$

We require that (4) should be an *isomorphism* over  $s_q^{-1}(0)$ .

Here Definition 2.3(e) replaces the notion in [11, Def. 5.6], [10, Def. A1.14] that a Kuranishi structure *has a tangent bundle*.

**Definition 2.4.** A *germ of Kuranishi neighbourhoods* at  $p \in X$  is an equivalence class of Kuranishi neighbourhoods  $(V_p, E_p, s_p, \psi_p)$  of  $p$ , using the notion of equivalence in Definition 2.2. Suppose  $(V_p, E_p, s_p, \psi_p)$  lies in such a germ. Then for any open neighbourhood  $U_p$  of  $\psi_p^{-1}(p)$  in  $V_p$ ,  $(U_p, E_p|_{U_p}, s_p|_{U_p}, \psi_p|_{U_p})$  also lies in the germ. As a shorthand, we say that some condition on the germ *holds for sufficiently small*  $(V_p, \dots, \psi_p)$  if whenever  $(V_p, \dots, \psi_p)$  lies in the germ, the condition holds for  $(U_p, \dots, \psi_p|_{U_p})$  for all sufficiently small  $U_p$  as above.

A *Kuranishi structure*  $\kappa$  on  $X$  assigns a germ of Kuranishi neighbourhoods for each  $p \in X$  and a *germ of coordinate changes* between them in the following sense: for each  $p \in X$ , for all sufficiently small  $(V_p, \dots, \psi_p)$  in the germ at  $p$ , for all  $q \in \text{Im } \psi_p$ , and for all sufficiently small  $(V_q, \dots, \psi_q)$  in the germ at  $q$ , we are given a coordinate change  $(\phi_{pq}, \hat{\phi}_{pq})$  from  $(V_q, \dots, \psi_q)$  to  $(V_p, \dots, \psi_p)$ . These coordinate changes should be compatible with equivalence in the germs at  $p, q$  in the obvious way, and satisfy:



- (i)  $\dim V_p - \text{rank } E_p$  is independent of  $p$ ; and
- (ii) if  $q \in \text{Im } \psi_p$  and  $r \in \text{Im } \psi_q$  then  $\phi_{pq} \circ \phi_{qr} = \phi_{pr}$  and  $\hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \hat{\phi}_{pr}$ .

We call  $\text{vdim } X = \dim V_p - \text{rank } E_p$  the *virtual dimension* of the Kuranishi structure. A *Kuranishi space*  $(X, \kappa)$  is a topological space  $X$  with a Kuranishi structure  $\kappa$ . Usually we refer to  $X$  as the Kuranishi space, suppressing  $\kappa$ .

Loosely speaking, the above definitions mean that a Kuranishi space is locally modelled on the zeroes of a smooth section of an orbifold vector bundle over an orbifold. Moduli spaces of  $J$ -holomorphic curves in Symplectic Geometry can be given Kuranishi structures in a natural way, as in [10, 11].

### 2.3 Strongly smooth maps and strong submersions

In [12, Def. 2.24] we define *strongly smooth maps*  $\mathbf{f} : X \rightarrow Y$ , for  $Y$  an orbifold.

**Definition 2.5.** Let  $X$  be a Kuranishi space, and  $Y$  a smooth orbifold. A *strongly smooth map*  $\mathbf{f} : X \rightarrow Y$  consists of, for all  $p \in X$  and all sufficiently small  $(V_p, E_p, s_p, \psi_p)$  in the germ of Kuranishi neighbourhoods at  $p$ , a choice of smooth map  $f_p : V_p \rightarrow Y$ , such that for all  $q \in \text{Im } \psi_p$  and sufficiently small  $(V_q, \dots, \psi_q)$  in the germ at  $q$  with coordinate change  $(\phi_{pq}, \hat{\phi}_{pq})$  from  $(V_q, \dots, \psi_q)$  to  $(V_p, \dots, \psi_p)$  in the germ of coordinate changes, we have  $f_p \circ \phi_{pq} = f_q$ . Then  $\mathbf{f}$  induces a continuous map  $f : X \rightarrow Y$  in the obvious way.

We call  $\mathbf{f}$  a *strong submersion* if all the  $f_p$  are submersions, that is, the maps  $df_p : TV_p \rightarrow f_p^*(TY)$  are surjective, and also when  $V_p$  has boundary or corners,  $f_p|_{\partial V_p} : \partial V_p \rightarrow Y$  is a submersion, and the restriction of  $f_p$  to each codimension  $k$  corner is a submersion for all  $k$ .

There is also [12, Def. 2.25] a definition of strongly smooth maps  $\mathbf{f} : X \rightarrow Y$  for  $X, Y$  Kuranishi spaces, which we will not give. A *strong diffeomorphism*  $\mathbf{f} : X \rightarrow Y$  is a strongly smooth map with a strongly smooth inverse. It is the natural notion of isomorphism of Kuranishi spaces.

### 2.4 Boundaries of Kuranishi spaces

We define the *boundary*  $\partial X$  of a Kuranishi space  $X$ , which is itself a Kuranishi space of dimension  $\text{vdim } X - 1$ .

**Definition 2.6.** Let  $X$  be a Kuranishi space. We shall define a Kuranishi space  $\partial X$  called the *boundary* of  $X$ . The points of  $\partial X$  are equivalence classes  $[p, (V_p, \dots, \psi_p), B]$  of triples  $(p, (V_p, \dots, \psi_p), B)$ , where  $p \in X$ ,  $(V_p, \dots, \psi_p)$  lies in the germ of Kuranishi neighbourhoods at  $p$ , and  $B$  is a local boundary component of  $V_p$  at  $\psi_p^{-1}(p)$ . Two triples  $(p, (V_p, \dots, \psi_p), B), (q, (\tilde{V}_q, \dots, \tilde{\psi}_q), C)$  are *equivalent* if  $p = q$ , and the Kuranishi neighbourhoods  $(V_p, \dots, \psi_p), (\tilde{V}_q, \dots, \tilde{\psi}_q)$  are equivalent so that we are given an isomorphism  $(\alpha, \hat{\alpha}) : (U_p, \dots, \psi_p|_{U_p}) \rightarrow (\tilde{U}_q, \dots, \tilde{\psi}_q|_{\tilde{U}_q})$  for open  $\psi_p^{-1}(p) \in U_p \subseteq V_p$  and  $\tilde{\psi}_q^{-1}(q) \in \tilde{U}_q \subseteq \tilde{V}_q$ , and  $\alpha_*(B) = C$  near  $\tilde{\psi}_q^{-1}(q)$ .

We can define a unique natural topology and Kuranishi structure on  $\partial X$ , such that  $(\partial V_p, E_p|_{\partial V_p}, s_p|_{\partial V_p}, \psi'_p)$  is a Kuranishi neighbourhood on  $\partial X$  for each Kuranishi neighbourhood  $(V_p, \dots, \psi_p)$  on  $X$ , where  $\psi'_p : (s_p|_{\partial V_p})^{-1}(0) \rightarrow \partial X$  is given by  $\psi'_p : (q, B) \mapsto [\psi_p(q), (V_p, \dots, \psi_p), B]$  for  $(q, B) \in \partial V_p$  with  $s_p(q) = 0$ . Then  $\text{vdim}(\partial X) = \text{vdim } X - 1$ , and  $\partial X$  is compact if  $X$  is compact.

In §2.1 we explained that if  $X$  is a manifold with (g-)corners then there is a natural involution  $\sigma : \partial^2 X \rightarrow \partial^2 X$ . The same construction works for orbifolds, and for Kuranishi spaces. That is, if  $X$  is a Kuranishi space then as in [12, §2.6] there is a natural strong diffeomorphism  $\sigma : \partial^2 X \rightarrow \partial^2 X$  with  $\sigma^2 = \text{id}_X$ . If  $X$  is oriented as in §2.6 below then  $\sigma$  is orientation-reversing.

## 2.5 Fibre products of Kuranishi spaces

We can define *fibre products* of Kuranishi spaces [12, Def. 2.28], as for fibre products of manifolds and orbifolds in §2.1.

**Definition 2.7.** Let  $X, X'$  be Kuranishi spaces,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$ ,  $\mathbf{f}' : X' \rightarrow Y$  be strongly smooth maps inducing continuous maps  $f : X \rightarrow Y$  and  $f' : X' \rightarrow Y$ . Suppose at least one of  $\mathbf{f}, \mathbf{f}'$  is a strong submersion. We shall define the *fibre product*  $X \times_Y X'$  or  $X \times_{\mathbf{f}, Y, \mathbf{f}'} X'$ , a Kuranishi space. As a set, the underlying topological space  $X \times_Y X'$  is given by (3).

Let  $p \in X$ ,  $p' \in X'$  and  $q \in Y$  with  $f(p) = q = f'(p')$ . Let  $(V_p, E_p, s_p, \psi_p)$ ,  $(V'_{p'}, E'_{p'}, s'_{p'}, \psi'_{p'})$  be sufficiently small Kuranishi neighbourhoods in the germs at  $p, p'$  in  $X, X'$ , and  $f_p : V_p \rightarrow Y$ ,  $f'_{p'} : V'_{p'} \rightarrow Y$  be smooth maps in the germs of  $\mathbf{f}, \mathbf{f}'$  at  $p, p'$  respectively. Define a Kuranishi neighbourhood on  $X \times_Y X'$  by

$$\begin{aligned} & (V_p \times_{f_p, Y, f'_{p'}} V'_{p'}, \pi_{V_p}^*(E_p) \oplus \pi_{V'_{p'}}^*(E'_{p'}), \\ & (s_p \circ \pi_{V_p}) \oplus (s'_{p'} \circ \pi_{V'_{p'}}), (\psi_p \circ \pi_{V_p}) \times (\psi'_{p'} \circ \pi_{V'_{p'}}) \times \chi_{pp'}). \end{aligned} \quad (5)$$

Here  $V_p \times_{f_p, Y, f'_{p'}} V'_{p'}$  is the fibre product of orbifolds, and  $\pi_{V_p}, \pi_{V'_{p'}}$  are the projections from  $V_p \times_Y V'_{p'}$  to  $V_p, V'_{p'}$ . The final term  $\chi_{pp'}$  in (5) maps the biquotient terms in (3) for  $V_p \times_Y V'_{p'}$  to the same terms in (3) for the set  $X \times_Y X'$ . Coordinate changes between Kuranishi neighbourhoods in  $X, X'$  induce coordinate changes between neighbourhoods (5). So the systems of germs of Kuranishi neighbourhoods and coordinate changes on  $X, X'$  induce such systems on  $X \times_Y X'$ . This gives a *Kuranishi structure* on  $X \times_Y X'$ , making it into a *Kuranishi space*. Clearly  $\text{vdim}(X \times_Y X') = \text{vdim } X + \text{vdim } X' - \dim Y$ , and  $X \times_Y X'$  is compact if  $X, X'$  are compact.

## 2.6 Orientations and orientation conventions

In [12, §2.7] we define *orientations* on Kuranishi spaces. Our definition is basically equivalent to Fukaya et al. [10, Def. A1.17], noting that our Kuranishi spaces correspond to their Kuranishi spaces with a tangent bundle.

**Definition 2.8.** Let  $X$  be a Kuranishi space. An *orientation* on  $X$  assigns, for all  $p \in X$  and all sufficiently small Kuranishi neighbourhoods  $(V_p, E_p, s_p, \psi_p)$  in

the germ at  $p$ , orientations on the fibres of the orbibundle  $TV_p \oplus E_p$  varying continuously over  $V_p$ . These must be compatible with coordinate changes, in the following sense. Let  $q \in \text{Im } \psi_p$ ,  $(V_q, \dots, \psi_q)$  be sufficiently small in the germ at  $q$ , and  $(\phi_{pq}, \hat{\phi}_{pq})$  be the coordinate change from  $(V_q, \dots, \psi_q)$  to  $(V_p, \dots, \psi_p)$  in the germ. Define  $d\hat{s}_p$  near  $s_q^{-1}(0) \subseteq V_q$  as in (4).

Locally on  $V_q$ , choose any orientation for the fibres of  $\phi_{pq}^*(TV_p)/(\text{d}\phi_{pq})(TV_q)$ , and let  $\phi_{pq}^*(E_p)/\hat{\phi}_{pq}(E_q)$  have the orientation induced from this by the isomorphism  $d\hat{s}_p$  in (4). These induce an orientation on  $\frac{\phi_{pq}^*(TV_p)}{(\text{d}\phi_{pq})(TV_q)} \oplus \frac{\phi_{pq}^*(E_p)}{\hat{\phi}_{pq}(E_q)}$ , which is independent of the choice for  $\phi_{pq}^*(TV_p)/(\text{d}\phi_{pq})(TV_q)$ . Thus, these local choices induce a natural orientation on the orbibundle  $\frac{\phi_{pq}^*(TV_p)}{(\text{d}\phi_{pq})(TV_q)} \oplus \frac{\phi_{pq}^*(E_p)}{\hat{\phi}_{pq}(E_q)}$  near  $s_q^{-1}(0)$ . We require that in oriented orbibundles over  $V_q$  near  $s_q^{-1}(0)$ , we have

$$\begin{aligned} \phi_{pq}^*[TV_p \oplus E_p] \cong (-1)^{\dim V_q(\dim V_p - \dim V_q)} [TV_q \oplus E_q] \oplus \\ \left[ \frac{\phi_{pq}^*(TV_p)}{(\text{d}\phi_{pq})(TV_q)} \oplus \frac{\phi_{pq}^*(E_p)}{\hat{\phi}_{pq}(E_q)} \right], \end{aligned} \quad (6)$$

where  $TV_p \oplus E_p$  and  $TV_q \oplus E_q$  have the orientations assigned by the orientation on  $X$ . An *oriented Kuranishi space* is a Kuranishi space with an orientation.

Suppose  $X, X'$  are oriented Kuranishi spaces,  $Y$  is an oriented orbifold, and  $f : X \rightarrow Y$ ,  $f' : X' \rightarrow Y$  are strong submersions. Then by §2.4–§2.5 we have Kuranishi spaces  $\partial X$  and  $X \times_Y X'$ . These can also be given orientations in a natural way. We shall follow the orientation conventions of Fukaya et al. [10, §45].

**Convention 2.9.** First, our conventions for manifolds:

- (a) Let  $X$  be an oriented manifold with boundary  $\partial X$ . Then we define the orientation on  $\partial X$  such that  $TX|_{\partial X} = \mathbb{R}_{\text{out}} \oplus T(\partial X)$  is an isomorphism of oriented vector spaces, where  $\mathbb{R}_{\text{out}}$  is oriented by an outward-pointing normal vector to  $\partial X$ .
- (b) Let  $X, X', Y$  be oriented manifolds, and  $f : X \rightarrow Y$ ,  $f' : X' \rightarrow Y$  be submersions. Then  $\text{d}f : TX \rightarrow f^*(TY)$  and  $\text{d}f' : TX' \rightarrow (f')^*(TY)$  are surjective maps of vector bundles over  $X, X'$ . Choosing Riemannian metrics on  $X, X'$  and identifying the orthogonal complement of  $\text{Ker } \text{d}f$  in  $TX$  with the image  $f^*(TY)$  of  $\text{d}f$ , and similarly for  $f'$ , we have isomorphisms of vector bundles over  $X, X'$ :

$$TX \cong \text{Ker } \text{d}f \oplus f^*(TY) \quad \text{and} \quad TX' \cong (f')^*(TY) \oplus \text{Ker } \text{d}f'. \quad (7)$$

Define orientations on the fibres of  $\text{Ker } \text{d}f$ ,  $\text{Ker } \text{d}f'$  over  $X, X'$  such that (7) are isomorphisms of oriented vector bundles, where  $TX, TX'$  are oriented by the orientations on  $X, X'$ , and  $f^*(TY), (f')^*(TY)$  by the orientation on  $Y$ . Then we define the orientation on  $X \times_Y X'$  so that

$$T(X \times_Y X') \cong \pi_X^*(\text{Ker } \text{d}f) \oplus (f \circ \pi_X)^*(TY) \oplus \pi_{X'}^*(\text{Ker } \text{d}f')$$

is an isomorphism of oriented vector bundles. Here  $\pi_X : X \times_Y X' \rightarrow X$  and  $\pi_{X'} : X \times_Y X' \rightarrow X'$  are the natural projections, and  $f \circ \pi_X \equiv f' \circ \pi_{X'}$ .

These extend immediately to orbifolds. They also extend to the Kuranishi space versions in Definitions 2.6 and 2.7; for Definition 2.7 they are described in [10, Conv. 45.1(4)]. An algorithm to deduce Kuranishi space orientation conventions from manifold ones is described in [12, §2.7].

If  $X$  is an oriented Kuranishi space, we often write  $-X$  for the same Kuranishi space with the opposite orientation. Here is [12, Prop. 2.31], largely taken from Fukaya et al. [10, Lem. 45.3].

**Proposition 2.10.** *Let  $X_1, X_2, \dots$  be oriented Kuranishi spaces,  $Y, Y_1, \dots$  be oriented orbifolds, and  $\mathbf{f}_1 : X_1 \rightarrow Y, \dots$  be strongly smooth maps, with at least one strong submersion in each fibre product below. Then the following hold, in oriented Kuranishi spaces:*

(a) *If  $\partial Y = \emptyset$ , for  $\mathbf{f}_1 : X_1 \rightarrow Y$  and  $\mathbf{f}_2 : X_2 \rightarrow Y$  we have*

$$\begin{aligned} \partial(X_1 \times_Y X_2) &= (\partial X_1) \times_Y X_2 \amalg (-1)^{\text{vdim } X_1 + \dim Y} X_1 \times_Y (\partial X_2) \\ \text{and } X_1 \times_Y X_2 &= (-1)^{(\text{vdim } X_1 - \dim Y)(\text{vdim } X_2 - \dim Y)} X_2 \times_Y X_1. \end{aligned} \quad (8)$$

(b) *For  $\mathbf{f}_1 : X_1 \rightarrow Y_1$ ,  $\mathbf{f}_2 : X_2 \rightarrow Y_1 \times Y_2$  and  $\mathbf{f}_3 : X_3 \rightarrow Y_2$ , we have*

$$(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 = X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3). \quad (9)$$

(c) *For  $\mathbf{f}_1 : X_1 \rightarrow Y_1 \times Y_2$ ,  $\mathbf{f}_2 : X_2 \rightarrow Y_1$  and  $\mathbf{f}_3 : X_3 \rightarrow Y_2$ , we have*

$$X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3) = (-1)^{\dim Y_2 (\dim Y_1 + \text{vdim } X_2)} (X_1 \times_{Y_1} X_2) \times_{Y_2} X_3. \quad (10)$$

## 2.7 Coorientations

To define Kuranishi cohomology in §4 we will also need a notion of *relative orientation* for (strong) submersions. We call this a *coorientation*, [12, §2.8].

**Definition 2.11.** Let  $X, Y$  be orbifolds, and  $f : X \rightarrow Y$  a submersion. A *coorientation* for  $(X, f)$  is a choice of orientations on the fibres of the vector bundle  $\text{Ker } df$  over  $X$  which vary continuously over  $X$ . Here  $df : TX \rightarrow f^*(TY)$  is the derivative of  $f$ , a morphism of vector bundles, which is surjective as  $f$  is a submersion. Thus  $\text{Ker } df$  is a vector bundle over  $X$ , of rank  $\dim X - \dim Y$ .

Now let  $X$  be a Kuranishi space,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$  a strong submersion. A *coorientation* for  $(X, \mathbf{f})$  assigns, for all  $p \in X$  and all sufficiently small Kuranishi neighbourhoods  $(V_p, E_p, s_p, \psi_p)$  in the germ at  $p$  with submersion  $f_p : V_p \rightarrow Y$  representing  $\mathbf{f}$ , orientations on the fibres of the orbibundle  $\text{Ker } df_p \oplus E_p$  varying continuously over  $V_p$ , where  $df_p : TV_p \rightarrow f_p^*(TY)$  is the (surjective) derivative of  $f_p$ .

These must be compatible with coordinate changes, in the following sense. Let  $q \in \text{Im } \psi_p$ ,  $(V_q, \dots, \psi_q)$  be sufficiently small in the germ at  $q$ , let  $f_q : V_q \rightarrow Y$  represent  $\mathbf{f}$ , and  $(\phi_{pq}, \hat{\phi}_{pq})$  be the coordinate change from  $(V_q, \dots, \psi_q)$  to  $(V_p, \dots, \psi_p)$  in the germ. Then we require that in oriented orbibundles over  $V_q$  near  $s_q^{-1}(0)$ , we have

$$\phi_{pq}^* [\text{Ker } df_p \oplus E_p] \cong (-1)^{\dim V_q(\dim V_p - \dim V_q)} [\text{Ker } df_q \oplus E_q] \oplus \left[ \frac{\phi_{pq}^*(TV_p)}{(\text{d}\phi_{pq})(TV_q)} \oplus \frac{\phi_{pq}^*(E_p)}{\phi_{pq}(E_q)} \right], \quad (11)$$

by analogy with (6), where  $\frac{\phi_{pq}^*(TV_p)}{(\text{d}\phi_{pq})(TV_q)} \oplus \frac{\phi_{pq}^*(E_p)}{\phi_{pq}(E_q)}$  is oriented as in Definition 2.8.

Suppose now that  $Y$  is oriented. Then an orientation on  $X$  is equivalent to a coorientation for  $(X, \mathbf{f})$ , since for all  $p$ ,  $(V_p, \dots, \psi_p)$ ,  $f_p$  as above, the isomorphism  $TV_p \cong f_p^*(TY) \oplus \text{Ker } df_p$  induces isomorphisms of orbibundles over  $V_p$ :

$$(TV_p \oplus E_p) \cong f_p^*(TY) \oplus (\text{Ker } df_p \oplus E_p). \quad (12)$$

There is a 1-1 correspondence between orientations on  $X$  and coorientations for  $(X, \mathbf{f})$  such that (12) holds in oriented orbibundles, where  $TV_p \oplus E_p$  is oriented by the orientation on  $X$ , and  $\text{Ker } df_p \oplus E_p$  by the coorientation for  $(X, \mathbf{f})$ , and  $f_p^*(TY)$  by the orientation on  $Y$ . Taking the direct sum of  $f_p^*(TY)$  with (11) and using (12) yields (6), so this is compatible with coordinate changes.

In [12, Conv. 2.33] we give our conventions for coorientations of boundaries and fibre products. These correspond to Convention 2.9 under the 1-1 correspondence between orientations on  $X$  and coorientations for  $(X, \mathbf{f})$  above when  $Y$  is oriented. So the analogue of Proposition 2.10 holds for coorientations. In particular, for strong submersions  $\mathbf{f}_a : X_a \rightarrow Y$  with  $(X_a, \mathbf{f}_a)$  cooriented for  $a = 1, 2, 3$ , taking  $\partial Y = \emptyset$  in (13), we have

$$(\partial(X_1 \times_Y X_2), \pi_Y) \cong ((\partial X_1) \times_Y X_2, \pi_Y) \amalg (-1)^{\text{vdim } X_1 + \dim Y} (X_1 \times_Y (\partial X_2), \pi_Y), \quad (13)$$

$$(X_1 \times_Y X_2, \pi_Y) \cong (-1)^{(\text{vdim } X_1 - \dim Y)(\text{vdim } X_2 - \dim Y)} (X_2 \times_Y X_1, \pi_Y), \quad (14)$$

$$((X_1 \times_Y X_2) \times_Y X_3, \pi_Y) \cong (X_1 \times_Y (X_2 \times_Y X_3), \pi_Y). \quad (15)$$

Similarly, if  $X_1$  is oriented,  $\mathbf{f}_1 : X_1 \rightarrow Y$  is strongly smooth,  $\mathbf{f}_2 : X_2 \rightarrow Y$  is a cooriented strong submersion, and  $\partial Y = \emptyset$  then

$$\partial(X_1 \times_Y X_2) \cong ((\partial X_1) \times_Y X_2) \amalg (-1)^{\text{vdim } X_1 + \dim Y} (X_1 \times_Y (\partial X_2)) \quad (16)$$

in oriented Kuranishi spaces.

### 3 Kuranishi homology

*Kuranishi homology* [12, §4] is a homology theory of orbifolds  $Y$  in which the chains are isomorphism classes  $[X, \mathbf{f}, \mathbf{G}]$ , where  $X$  is a compact, oriented Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is strongly smooth, and  $\mathbf{G}$  is some extra data called *gauge-fixing data*. It is isomorphic to singular homology  $H_*^{\text{si}}(Y; R)$ .

### 3.1 Gauge-fixing data

Let  $X$  be a compact Kuranishi space,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$  a strongly smooth map. A key ingredient in the definition of Kuranishi homology in [12] is the idea of *gauge fixing data*  $\mathbf{G}$  for  $(X, \mathbf{f})$  studied in [12, §6]. Define  $P = \coprod_{k=0}^{\infty} \mathbb{R}^k / S_k$ , where the symmetric group  $S_k$  acts on  $\mathbb{R}^k$  by permuting the coordinates  $x_1, \dots, x_k$ . For  $n = 0, 1, 2, \dots$ , define  $P_n \subset P$  by  $P_n = \coprod_{k=0}^n \mathbb{R}^k / S_k$ . Gauge-fixing data  $\mathbf{G}$  for  $(X, \mathbf{f})$  consists of a cover of  $X$  by Kuranishi neighbourhoods  $(V^i, E^i, s^i, \psi^i)$  for  $i$  in a finite indexing set  $I$ , together with smooth maps  $f^i : V^i \rightarrow Y$  representing  $\mathbf{f}$  and maps  $G^i : E^i \rightarrow P_n \subset P$  for some  $n \gg 0$ , and continuous partitions of unity  $\eta_i : X \rightarrow [0, 1]$  and  $\eta_i^j : V^j \rightarrow [0, 1]$ , satisfying many conditions. One important condition, responsible for Theorem 3.1(b) below, is that each  $G^i : E^i \rightarrow P$  should be a *finite* map, that is,  $(G^i)^{-1}(p)$  is finitely many points for all  $p \in P$ .

Users of Kuranishi homology do not need to know exactly what gauge-fixing data is, so we will not define it. Here are the important properties of gauge-fixing data, which are proved in [12, §3].

**Theorem 3.1.** *Consider pairs  $(X, \mathbf{f})$ , where  $X$  is a compact Kuranishi space,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$  a strongly smooth map. In [12, §3.1] we define **gauge-fixing data**  $\mathbf{G}$  for such pairs  $(X, \mathbf{f})$ , with the following properties:*

- (a) *Every pair  $(X, \mathbf{f})$  admits a (nonunique) choice of gauge-fixing data  $\mathbf{G}$ . If  $\Gamma \subseteq \text{Aut}(X, \mathbf{f})$  is a finite subgroup then we can choose  $\mathbf{G}$  to be  $\Gamma$ -invariant.*
- (b) *For all pairs  $(X, \mathbf{f})$  and choices of gauge-fixing data  $\mathbf{G}$  for  $(X, \mathbf{f})$ , the automorphism group  $\text{Aut}(X, \mathbf{f}, \mathbf{G})$  of isomorphisms  $(\mathbf{a}, \mathbf{b}) : (X, \mathbf{f}, \mathbf{G}) \rightarrow (X, \mathbf{f}, \mathbf{G})$  is finite.*
- (c) *Suppose  $\mathbf{G}$  is gauge-fixing data for  $(X, \mathbf{f})$  and  $\Gamma$  is a finite group acting on  $(X, \mathbf{f}, \mathbf{G})$  by isomorphisms. Then we can form the quotient  $\tilde{X} = X/\Gamma$ , a compact Kuranishi space, with projection  $\pi : X \rightarrow \tilde{X}$ , and  $\mathbf{f}$  pushes down to  $\tilde{\mathbf{f}} : \tilde{X} \rightarrow Y$  with  $\mathbf{f} = \tilde{\mathbf{f}} \circ \pi$ . As in [12, §3.4], we can define gauge-fixing data  $\tilde{\mathbf{G}}$  for  $(\tilde{X}, \tilde{\mathbf{f}})$ , which is the natural push down  $\pi_*(\mathbf{G})$  of  $\mathbf{G}$  to  $\tilde{X}$ .*
- (d) *If  $\mathbf{G}$  is gauge-fixing data for  $(X, \mathbf{f})$ , it has a restriction  $\mathbf{G}|_{\partial X}$  defined in [12, §3.5], which is gauge-fixing data for  $(\partial X, \mathbf{f}|_{\partial X})$ .*
- (e) *Let  $X$  be a compact, oriented Kuranishi space with corners (not  $g$ -corners),  $\mathbf{f} : X \rightarrow Y$  be strongly smooth, and  $\sigma : \partial^2 X \rightarrow \partial^2 X$  be the natural involution described in §2.4. Suppose  $\mathbf{H}$  is gauge-fixing data for  $(\partial X, \mathbf{f}|_{\partial X})$ . Then there exists gauge-fixing data  $\mathbf{G}$  for  $(X, \mathbf{f})$  with  $\mathbf{G}|_{\partial X} = \mathbf{H}$  if and only if  $\mathbf{H}|_{\partial^2 X}$  is invariant under  $\sigma$ . If also  $\Gamma$  is a finite subgroup of  $\text{Aut}(X, \mathbf{f})$ , and  $\mathbf{H}$  is invariant under  $\Gamma|_{\partial X}$ , then we can choose  $\mathbf{G}$  to be  $\Gamma$ -invariant.*
- (f) *Let  $Y, Z$  be orbifolds, and  $h : Y \rightarrow Z$  a smooth map. Suppose  $X$  is a compact Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is strongly smooth, and  $\mathbf{G}$  is gauge-fixing data for  $(X, \mathbf{f})$ . Then as in [12, §3.7], we can define gauge-fixing data  $h_*(\mathbf{G})$  for  $(X, h \circ \mathbf{f})$ . It satisfies  $(g \circ h)_*(\mathbf{G}) = g_*(h_*(\mathbf{G}))$ .*

In order to define a working homology theory, perhaps the most important property is Theorem 3.1(b). In [12, §4.9] we show that if we define *naïve Kuranishi (co)homology*  $KH_*^{\text{na}}, KH_*^{\text{na}}(Y; R)$  as in §3.2 and §4.2 but omitting all (co-)gauge-fixing data, then  $KH_*^{\text{na}}(Y; R) = 0 = KH_*^{\text{na}}(Y; R)$  for all orbifolds  $Y$  and  $\mathbb{Q}$ -algebras  $R$ . For compact  $Y$ , we do this by constructing an explicit cochain whose boundary is the identity cocycle  $[Y, \text{id}_Y]$  in  $KC_{\text{na}}^*(Y; R)$ . This explicit cochain involves a cocycle  $[X, \mathbf{f}]$  whose automorphism group  $\text{Aut}(X, \mathbf{f})$  is infinite. Including (co-)gauge-fixing data prevents this from happening, as it ensures that all automorphism groups  $\text{Aut}(X, \mathbf{f}, \mathbf{G})$  are finite.

### 3.2 Kuranishi homology

We can now define the *Kuranishi homology* of an orbifold, [12, §4.2].

**Definition 3.2.** Let  $Y$  be an orbifold. Consider triples  $(X, \mathbf{f}, \mathbf{G})$ , where  $X$  is a compact, oriented Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is strongly smooth, and  $\mathbf{G}$  is gauge-fixing data for  $(X, \mathbf{f})$ . Write  $[X, \mathbf{f}, \mathbf{G}]$  for the isomorphism class of  $(X, \mathbf{f}, \mathbf{G})$  under isomorphisms  $(\mathbf{a}, \mathbf{b}) : (X, \mathbf{f}, \mathbf{G}) \rightarrow (\tilde{X}, \tilde{\mathbf{f}}, \tilde{\mathbf{G}})$ , where  $\mathbf{a}$  must identify the orientations of  $X, \tilde{X}$ , and  $\mathbf{b}$  lifts  $\mathbf{a}$  to the Kuranishi neighbourhoods  $(V^i, \dots, \psi^i), (\tilde{V}^i, \dots, \tilde{\psi}^i)$  in  $\mathbf{G}, \tilde{\mathbf{G}}$ .

Let  $R$  be a  $\mathbb{Q}$ -algebra, for instance  $\mathbb{Q}, \mathbb{R}$  or  $\mathbb{C}$ . For each  $k \in \mathbb{Z}$ , define  $KC_k(Y; R)$  to be the  $R$ -module of finite  $R$ -linear combinations of isomorphism classes  $[X, \mathbf{f}, \mathbf{G}]$  for which  $\text{vdim } X = k$ , with the relations:

- (i) Let  $[X, \mathbf{f}, \mathbf{G}]$  be an isomorphism class, and write  $-X$  for  $X$  with the opposite orientation. Then in  $KC_k(Y; R)$  we have

$$[X, \mathbf{f}, \mathbf{G}] + [-X, \mathbf{f}, \mathbf{G}] = 0.$$

- (ii) Let  $[X, \mathbf{f}, \mathbf{G}]$  be an isomorphism class. Suppose that  $X$  may be written as a disjoint union  $X = X_+ \amalg X_-$  of compact, oriented Kuranishi spaces, and that for each Kuranishi neighbourhood  $(V^i, \dots, \psi^i)$  for  $i \in I$  in  $\mathbf{G}$  we may write  $V^i = V_+^i \amalg V_-^i$  for open and closed subsets  $V_{\pm}^i$  of  $V^i$ , such that  $\text{Im } \psi^i|_{V_+^i} \subseteq X_+$  and  $\text{Im } \psi^i|_{V_-^i} \subseteq X_-$ . Then we may define gauge-fixing data  $\mathbf{G}|_{X_{\pm}}$  for  $(X_{\pm}, \mathbf{f}|_{X_{\pm}})$ , with Kuranishi neighbourhoods  $(V_{\pm}^i, E^i|_{V_{\pm}^i}, s^i|_{V_{\pm}^i}, \psi^i|_{V_{\pm}^i})$  for  $i \in I$  with  $V_{\pm}^i \neq \emptyset$ . In  $KC_k(Y; R)$  we have

$$[X, \mathbf{f}, \mathbf{G}] = [X_+, \mathbf{f}|_{X_+}, \mathbf{G}|_{X_+}] + [X_-, \mathbf{f}|_{X_-}, \mathbf{G}|_{X_-}].$$

- (iii) Let  $[X, \mathbf{f}, \mathbf{G}]$  be an isomorphism class, and suppose  $\Gamma$  is a finite group acting on  $(X, \mathbf{f}, \mathbf{G})$  by orientation-preserving automorphisms. Then  $\tilde{X} = X/\Gamma$  is a compact, oriented Kuranishi space, with a projection  $\pi : X \rightarrow \tilde{X}$ . As in Theorem 3.1(c),  $\mathbf{f}, \mathbf{G}$  push down to a strong submersion  $\pi_*(\mathbf{f}) = \tilde{\mathbf{f}} : \tilde{X} \rightarrow Y$  and gauge-fixing data  $\pi_*(\mathbf{G}) = \tilde{\mathbf{G}}$  for  $(\tilde{X}, \tilde{\mathbf{f}})$ . Then

$$[X/\Gamma, \pi_*(\mathbf{f}), \pi_*(\mathbf{G})] = \frac{1}{|\Gamma|} [X, \mathbf{f}, \mathbf{G}]$$

in  $KC_k(Y; R)$ . Elements of  $KC_k(Y; R)$  will be called *Kuranishi chains*.

Define the *boundary operator*  $\partial : KC_k(Y; R) \rightarrow KC_{k-1}(Y; R)$  by

$$\partial : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a, \mathbf{G}_a] \mapsto \sum_{a \in A} \rho_a [\partial X_a, \mathbf{f}_a|_{\partial X_a}, \mathbf{G}_a|_{\partial X_a}],$$

where  $A$  is a finite indexing set and  $\rho_a \in R$  for  $a \in A$ . This is a morphism of  $R$ -modules. Clearly,  $\partial$  takes each relation (i)–(iii) in  $KC_k(Y; R)$  to the corresponding relation in  $KC_{k-1}(Y; R)$ , and so  $\partial$  is well-defined.

Recall from §2.1 and §2.4 that if  $X$  is an oriented Kuranishi space then there is a natural orientation-reversing strong diffeomorphism  $\sigma : \partial^2 X \rightarrow \partial^2 X$ , with  $\sigma^2 = \text{id}_{\partial^2 X}$ . If  $[X, \mathbf{f}, \mathbf{G}]$  is an isomorphism class then this  $\sigma$  extends to an isomorphism  $(\sigma, \tau)$  of  $(\partial^2 X, \mathbf{f}|_{\partial^2 X}, \mathbf{G}|_{\partial^2 X})$ . So part (i) in  $KC_{k-2}(Y; R)$  yields

$$[\partial^2 X, \mathbf{f}|_{\partial^2 X}, \mathbf{G}|_{\partial^2 X}] + [\partial^2 X, \mathbf{f}|_{\partial^2 X}, \mathbf{G}|_{\partial^2 X}] = 0 \quad \text{in } KC_{k-2}(Y; R). \quad (17)$$

As  $R$  is a  $\mathbb{Q}$ -algebra we may multiply (17) by  $\frac{1}{2}$  to get  $[\partial^2 X, \mathbf{f}|_{\partial^2 X}, \mathbf{G}|_{\partial^2 X}] = 0$ . Therefore  $\partial \circ \partial = 0$  as a map  $KC_k(Y; R) \rightarrow KC_{k-2}(Y; R)$ .

Define the *Kuranishi homology group*  $KH_k(Y; R)$  of  $Y$  for  $k \in \mathbb{Z}$  to be

$$KH_k(Y; R) = \frac{\text{Ker}(\partial : KC_k(Y; R) \rightarrow KC_{k-1}(Y; R))}{\text{Im}(\partial : KC_{k+1}(Y; R) \rightarrow KC_k(Y; R))}.$$

Let  $Y, Z$  be orbifolds, and  $h : Y \rightarrow Z$  a smooth map. Define the *pushforward*  $h_* : KC_k(Y; R) \rightarrow KC_k(Z; R)$  for  $k \in \mathbb{Z}$  by

$$h_* : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a, \mathbf{G}_a] \mapsto \sum_{a \in A} \rho_a [X_a, h \circ \mathbf{f}_a, h_*(\mathbf{G}_a)],$$

with  $h_*(\mathbf{G}_a)$  as in Theorem 3.1(f). These take relations (i)–(iii) in  $KC_k(Y; R)$  to (i)–(iii) in  $KC_k(Z; R)$ , and so are well-defined. They satisfy  $h_* \circ \partial = \partial \circ h_*$ , so they induce morphisms of homology groups  $h_* : KH_k(Y; R) \rightarrow KH_k(Z; R)$ . Pushforward is functorial, that is,  $(g \circ h)_* = g_* \circ h_*$ , on chains and homology.

### 3.3 Singular homology and Kuranishi homology

Let  $Y$  be an orbifold, and  $R$  a  $\mathbb{Q}$ -algebra. Then we can define the *singular homology groups*  $H_k^{\text{si}}(Y; R)$ , as in Bredon [4, §IV]. Write  $C_k^{\text{si}}(Y; R)$  for the  $R$ -module spanned by *smooth singular  $k$ -simplices* in  $Y$ , which are smooth maps  $\sigma : \Delta_k \rightarrow Y$ , where  $\Delta_k$  is the  $k$ -simplex

$$\Delta_k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_i \geq 0, x_0 + \dots + x_k = 1\}.$$

As in [4, §IV.1], the boundary operator  $\partial : C_k^{\text{si}}(Y; R) \rightarrow C_{k-1}^{\text{si}}(Y; R)$  is given by

$$\partial : \sum_{a \in A} \rho_a \sigma_a \mapsto \sum_{a \in A} \sum_{j=0}^k (-1)^j \rho_a (\sigma_a \circ F_j^k),$$

where  $F_j^k : \Delta_{k-1} \rightarrow \Delta_k$ ,  $F_j^k : (x_0, \dots, x_{k-1}) \mapsto (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{k-1})$  for  $j = 0, \dots, k$ . Then  $\partial^2 = 0$ , and  $H_*^{\text{si}}(Y; R)$  is the homology of  $(C_k^{\text{si}}(Y; R), \partial)$ .



Define  $R$ -module morphisms  $C_k^{\text{si}}(Y; R) \rightarrow KC_k(Y; R)$  for  $k \geq 0$  by

$$\Pi_{\text{si}}^{\text{Kh}} : \sum_{a \in A} \rho_a \sigma_a \mapsto \sum_{a \in A} \rho_a [\Delta_k, \sigma_a, \mathbf{G}_{\Delta_k}],$$

where  $\mathbf{G}_{\Delta_k}$  is an explicit choice of gauge-fixing data for  $(\Delta_k, \sigma_a)$  given in [12, §4.3]. These satisfy  $\partial \circ \Pi_{\text{si}}^{\text{Kh}} = \Pi_{\text{si}}^{\text{Kh}} \circ \partial$ , and so induce  $R$ -module morphisms

$$\Pi_{\text{si}}^{\text{Kh}} : H_k^{\text{si}}(Y; R) \longrightarrow KH_k(Y; R). \quad (18)$$

Here is [12, Cor. 4.10], one of the main results of [12].

**Theorem 3.3.** *Let  $Y$  be an orbifold and  $R$  a  $\mathbb{Q}$ -algebra. Then  $\Pi_{\text{si}}^{\text{Kh}}$  in (18) is an isomorphism, so that  $KH_k(Y; R) \cong H_k^{\text{si}}(Y; R)$ , with  $KH_k(Y; R) = \{0\}$  when  $k < 0$ .*

The proof of Theorem 3.3 in [12, App. A–C] is very long and complex, taking up a third of [12]. The problem is to construct an inverse for  $\Pi_{\text{si}}^{\text{Kh}}$  in (18). This is related to Fukaya and Ono’s construction of *virtual cycles* for compact, oriented Kuranishi spaces without boundary in [11, §6], and uses some of the same ideas. But dealing with boundaries and corners of the Kuranishi spaces in Kuranishi chains, and the relations in the Kuranishi chain groups  $KC_*(Y; R)$ , increases the complexity by an order of magnitude.

The basic idea of the proof is to take a class  $\alpha \in KH_k(Y; R)$  and represent it by cycles  $\sum_{a \in A} \rho_a [X_a, \mathbf{f}_a, \mathbf{G}_a]$  with better and better properties, until eventually we represent it by a cycle in the image of  $\Pi_{\text{si}}^{\text{Kh}} : C_k^{\text{si}}(Y; R) \rightarrow KC_k(Y; R)$ , so showing that (18) is surjective. Here (somewhat oversimplified) are the main steps: firstly, by ‘cutting’ the  $X_a$  into small pieces  $X_{ac}$  for  $c \in C_a$ , we show we can represent  $\alpha$  by a cycle  $\sum_{a \in A} \sum_{c \in C_a} \rho_a [X_{ac}, \mathbf{f}_{ac}, \mathbf{G}_{ac}]$  such that  $(X_{ac}, \mathbf{f}_{ac}, \mathbf{G}_{ac})$  is the quotient of a triple  $(\tilde{X}_{ac}, \tilde{\mathbf{f}}_{ac}, \tilde{\mathbf{G}}_{ac})$  by a finite group  $\Gamma_{ac}$ , where  $\tilde{X}_{ac}$  has *trivial stabilizers*.

Thus by Definition 3.2(iii) we can represent  $\alpha$  by a cycle  $\sum_{a,c} \rho_a |\Gamma_{ac}|^{-1} [\tilde{X}_{ac}, \tilde{\mathbf{f}}_{ac}, \tilde{\mathbf{G}}_{ac}]$  involving only Kuranishi spaces  $\tilde{X}_{ac}$  with trivial stabilizers. Such spaces can be deformed to manifolds with  $g$ -corners (by single-valued perturbations, not multisections). So secondly, we show we can represent  $\alpha$  by a cycle  $\sum_{a,c} \rho_a |\Gamma_{ac}|^{-1} [\tilde{X}_{ac}, \tilde{f}_{ac}, \tilde{\mathbf{G}}_{ac}]$  in which the  $\tilde{X}_{ac}$  are manifolds, and  $\tilde{f}_{ac} : \tilde{X}_{ac} \rightarrow Y$  are smooth maps. Then thirdly, we triangulate the  $\tilde{X}_{ac}$  by simplices  $\Delta_k$ , and so prove that we can represent  $\alpha$  by a cycle in the image of  $\Pi_{\text{si}}^{\text{Kh}}$ . In this third step it is vital to work with manifolds *with  $g$ -corners*, as in §2.1, not just manifolds with corners, since otherwise we would not be able to construct the homology between  $\sum_{a,c} \rho_a |\Gamma_{ac}|^{-1} [\tilde{X}_{ac}, \tilde{f}_{ac}, \tilde{\mathbf{G}}_{ac}]$  and the singular cycle.

In the proof we use the fact that  $R$  is a  $\mathbb{Q}$ -algebra in two different ways. When we replace  $[X_{ac}, \mathbf{f}_{ac}, \mathbf{G}_{ac}]$  by  $|\Gamma_{ac}|^{-1} [\tilde{X}_{ac}, \tilde{\mathbf{f}}_{ac}, \tilde{\mathbf{G}}_{ac}]$  we must have  $|\Gamma_{ac}|^{-1} \in R$ , so we need  $\mathbb{Q} \subseteq R$ . And when we deform  $\tilde{X}_{ac}$  to manifolds  $\tilde{X}_{ac}$ , to make  $\sum_{a,c} \rho_a |\Gamma_{ac}|^{-1} [\tilde{X}_{ac}, \tilde{f}_{ac}, \tilde{\mathbf{G}}_{ac}]$  a cycle, we should ensure our perturbations are preserved by the automorphism groups  $\text{Aut}(\tilde{X}_{ac}, \tilde{\mathbf{f}}_{ac}, \tilde{\mathbf{G}}_{ac})$ . In fact this may not be possible, if  $\text{Aut}(\tilde{X}_{ac}, \tilde{\mathbf{f}}_{ac}, \tilde{\mathbf{G}}_{ac})$  has fixed points. So instead, we choose one perturbation  $\tilde{X}_{ac}$ , and then take the average of the images of this perturbation

under  $\text{Aut}(\dot{X}_{ac}, \dot{f}_{ac}, \dot{G}_{ac})$ . This requires us to divide by  $|\text{Aut}(\dot{X}_{ac}, \dot{f}_{ac}, \dot{G}_{ac})|$ , so again we need  $\mathbb{Q} \subseteq R$ . Also, for this step it is necessary that automorphism groups  $\text{Aut}(X, \mathbf{f}, \mathbf{G})$  should be *finite*, as in Theorem 3.1(b), and this was the reason for introducing gauge-fixing data.

The theorem means that in many problems, particularly areas of in Symplectic Geometry involving moduli spaces of  $J$ -holomorphic curves, we can use Kuranishi chains and homology instead of singular chains and homology, which can simplify proofs considerably, and also improve results.

## 4 Kuranishi cohomology

We now discuss the Poincaré dual theory of *Kuranishi cohomology*  $KH^*(Y; R)$ , which is isomorphic to compactly-supported cohomology  $H_{cs}^*(Y; R)$ . It is defined using a complex of Kuranishi cochains  $KC^*(Y; R)$  spanned by isomorphism classes  $[X, \mathbf{f}, \mathbf{C}]$  of triples  $(X, \mathbf{f}, \mathbf{C})$ , where  $X$  is a compact Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is a *cooriented strong submersion*, and  $\mathbf{C}$  is *co-gauge-fixing data*.

As is usual for cohomology, Kuranishi cohomology has an associative, supercommutative *cup product*  $\cup : KH^k(Y; R) \times KH^l(Y; R) \rightarrow KH^{k+l}(Y; R)$ , and there is also a *cap product*  $\cap : KH_k(Y; R) \times KH^l(Y; R) \rightarrow KH_{k-l}(Y; R)$  relating Kuranishi homology and Kuranishi (co)homology, which makes  $KH_*(Y; R)$  into a module over  $KH^*(Y; R)$ . More unusually, we can define  $\cup, \cap$  naturally on Kuranishi (co)chains, and  $\cup$  is associative and supercommutative on  $KC^*(Y; R)$ , and  $\cap$  makes  $KC_*(Y; R)$  into a module over  $KC^*(Y; R)$ .

### 4.1 Co-gauge-fixing data

Let  $X$  be a compact Kuranishi space,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$  a strong submersion. Kuranishi cohomology is based on the idea of *co-gauge-fixing data*  $\mathbf{C}$  for  $(X, \mathbf{f})$ . This is very similar to gauge-fixing data  $\mathbf{G}$  in §3.1, and consists of a finite cover of  $X$  by Kuranishi neighbourhoods  $(V^i, E^i, s^i, \psi^i)$  for  $i \in I$ , submersions  $f^i : V^i \rightarrow Y$  representing  $\mathbf{f}$  and maps  $C^i : E^i \rightarrow P_n \subset P$  for some  $n \gg 0$ , and partitions of unity  $\eta_i : X \rightarrow [0, 1]$  and  $\eta_i^j : V^j \rightarrow [0, 1]$ .

Here are the important properties of co-gauge-fixing data, which are proved in [12, §3]. Part (g) makes cup products work on Kuranishi cochains, and part (h) makes cap products work. It was difficult to find a definition of (co-)gauge-fixing data for which properties (a)–(h) all hold at once; a large part of the complexity of [12, §3] is due to the author's determination to ensure that cup products should be associative and supercommutative *at the cochain level*. This is not essential for a well-behaved (co)homology theory, but is extremely useful in the applications [2, 13, 14].

**Theorem 4.1.** *Consider pairs  $(X, \mathbf{f})$ , where  $X$  is a compact Kuranishi space,  $Y$  an orbifold, and  $\mathbf{f} : X \rightarrow Y$  a strong submersion. In [12, §3.1] we define **co-gauge-fixing data**  $\mathbf{C}$  for such pairs  $(X, \mathbf{f})$ . It satisfies the analogues of Theorem 3.1(a)–(e), and also:*

- (f) Let  $Y, Z$  be orbifolds, and  $h : Y \rightarrow Z$  a smooth, proper map. Suppose  $X$  is a compact Kuranishi space,  $\mathbf{f} : X \rightarrow Z$  is a strong submersion, and  $\mathbf{C}$  is co-gauge-fixing data for  $(X, \mathbf{f})$ . Then the fibre product  $Y \times_{h, Z, \mathbf{f}} X$  is a compact Kuranishi space, and  $\pi_Y : Y \times_Z X \rightarrow Y$  is a strong submersion. As in [12, §3.7], we can define co-gauge-fixing data  $h^*(\mathbf{C})$  for  $(Y \times_Z X, \pi_Y)$ . It satisfies  $(g \circ h)^*(\mathbf{C}) = h^*(g^*(\mathbf{C}))$ .
- (g) Let  $X_1, X_2, X_3$  be compact Kuranishi spaces,  $Y$  an orbifold,  $\mathbf{f}_i : X_i \rightarrow Y$  be strong submersions for  $i = 1, 2, 3$ , and  $\mathbf{C}_i$  be co-gauge-fixing data for  $(X_i, \mathbf{f}_i)$  for  $i = 1, 2, 3$ . Then [12, §3.8] defines co-gauge-fixing data  $\mathbf{C}_1 \times_Y \mathbf{C}_2$  for  $(X_1 \times_{\mathbf{f}_1, Y, \mathbf{f}_2} X_2, \pi_Y)$  from  $\mathbf{C}_1, \mathbf{C}_2$ .

This construction is **symmetric**, in that it yields isomorphic co-gauge-fixing data for  $(X_1 \times_Y X_2, \pi_Y)$  and  $(X_2 \times_Y X_1, \pi_Y)$  under the natural isomorphism  $X_1 \times_Y X_2 \cong X_2 \times_Y X_1$ . It is also **associative**, in that it yields isomorphic co-gauge-fixing data for  $((X_1 \times_Y X_2) \times_Y X_3, \pi_Y)$  and  $(X_1 \times_Y (X_2 \times_Y X_3), \pi_Y)$  under  $(X_1 \times_Y X_2) \times_Y X_3 \cong X_1 \times_Y (X_2 \times_Y X_3)$ .

These properties also have straightforward generalizations to multiple fibre products involving more than one orbifold  $Y$ , such as (9) and (10).

- (h) Let  $X_1, X_2$  be compact Kuranishi spaces,  $Y$  an orbifold,  $\mathbf{f}_1 : X_1 \rightarrow Y$  be strongly smooth,  $\mathbf{f}_2 : X_2 \rightarrow Y$  be a strong submersion,  $\mathbf{G}_1$  be gauge-fixing data for  $(X_1, \mathbf{f}_1)$ , and  $\mathbf{C}_2$  be co-gauge-fixing data for  $(X_2, \mathbf{f}_2)$ . Then [12, §3.8] defines gauge-fixing data  $\mathbf{G}_1 \times_Y \mathbf{C}_2$  for  $(X_1 \times_{\mathbf{f}_1, Y, \mathbf{f}_2} X_2, \pi_Y)$  from  $\mathbf{G}_1, \mathbf{C}_2$ . If also  $\mathbf{f}_3 : X_3 \rightarrow Y$  is a strong submersion and  $\mathbf{C}_3$  is co-gauge-fixing data for  $(X_3, \mathbf{f}_3)$  then the natural isomorphism  $(X_1 \times_Y X_2) \times_Y X_3 \cong X_1 \times_Y (X_2 \times_Y X_3)$  identifies  $(\mathbf{G}_1 \times_Y \mathbf{C}_2) \times_Y \mathbf{C}_3$  and  $\mathbf{G}_1 \times_Y (\mathbf{C}_2 \times_Y \mathbf{C}_3)$ .

## 4.2 Kuranishi cohomology

Here is our definition of Kuranishi cohomology [12, §4.4].

**Definition 4.2.** Let  $Y$  be an orbifold without boundary. Consider triples  $(X, \mathbf{f}, \mathbf{C})$ , where  $X$  is a compact Kuranishi space,  $\mathbf{f} : X \rightarrow Y$  is a strong submersion with  $(X, \mathbf{f})$  *cooriented*, as in §2.7, and  $\mathbf{C}$  is *co-gauge-fixing data* for  $(X, \mathbf{f})$ , as in §4.1. Write  $[X, \mathbf{f}, \mathbf{C}]$  for the isomorphism class of  $(X, \mathbf{f}, \mathbf{C})$  under isomorphisms  $(\mathbf{a}, \mathbf{b}) : (X, \mathbf{f}, \mathbf{C}) \rightarrow (\tilde{X}, \tilde{\mathbf{f}}, \tilde{\mathbf{C}})$ , where  $\mathbf{a}$  must identify the coorientations of  $(X, \mathbf{f})$ ,  $(\tilde{X}, \tilde{\mathbf{f}})$ , and  $\mathbf{b}$  lifts  $\mathbf{a}$  to the Kuranishi neighbourhoods  $(V^i, \dots, \psi^i), (\tilde{V}^i, \dots, \tilde{\psi}^i)$  in  $\mathbf{C}, \tilde{\mathbf{C}}$ .

Let  $R$  be a  $\mathbb{Q}$ -algebra. For  $k \in \mathbb{Z}$ , define  $KC^k(Y; R)$  to be the  $R$ -module of finite  $R$ -linear combinations of isomorphism classes  $[X, \mathbf{f}, \mathbf{C}]$  for which  $\text{vdim } X = \dim Y - k$ , with the analogues of relations Definition 3.2(i)–(iii), replacing gauge-fixing data  $\mathbf{G}$  by co-gauge-fixing data  $\mathbf{C}$ . Elements of  $KC^k(Y; R)$  are called *Kuranishi cochains*. Define  $d : KC^k(Y; R) \rightarrow KC^{k+1}(Y; R)$  by

$$d : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a, \mathbf{C}_a] \longmapsto \sum_{a \in A} \rho_a [\partial X_a, \mathbf{f}_a|_{\partial X_a}, \mathbf{C}_a|_{\partial X_a}]. \quad (19)$$

As in Definition 3.2 we have  $d \circ d = 0$ . Define the *Kuranishi cohomology groups*  $KH^k(Y; R)$  of  $Y$  for  $k \in \mathbb{Z}$  to be

$$KH^k(Y; R) = \frac{\text{Ker}(d : KC^k(Y; R) \rightarrow KC^{k+1}(Y; R))}{\text{Im}(d : KC^{k-1}(Y; R) \rightarrow KC^k(Y; R))}.$$

Let  $Y, Z$  be orbifolds without boundary, and  $h : Y \rightarrow Z$  be a smooth, proper map. Define the *pullback*  $h^* : KC^k(Z; R) \rightarrow KC^k(Y; R)$  by

$$h^* : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a, \mathbf{C}_a] \mapsto \sum_{a \in A} \rho_a [Y \times_{h, Z, \mathbf{f}_a} X_a, \pi_Y, h^*(\mathbf{C}_a)], \quad (20)$$

where  $h^*(\mathbf{C}_a)$  is as in Theorem 4.1(f), and the coorientation for  $(X_a, \mathbf{f}_a)$  pulls back to a natural coorientation for  $(Y \times_Z X_a, \pi_Y)$ . These  $h^* : KC^k(Z; R) \rightarrow KC^k(Y; R)$  satisfy  $h^* \circ d = d \circ h^*$ , so they induce morphisms of cohomology groups  $h^* : KH^k(Z; R) \rightarrow KH^k(Y; R)$ . Pullbacks are functorial, that is,  $(g \circ h)^* = h^* \circ g^*$ , on both cochains and cohomology.

Here for simplicity we restrict to orbifolds  $Y$  *without boundary*. When  $\partial Y \neq \emptyset$ , the definition of  $KH^*(Y; R)$  is more complicated [12, §4.5]: if  $\mathbf{f} : X \rightarrow Y$  is a strong submersion then we must split  $\partial X = \partial_+^{\mathbf{f}} X \amalg \partial_-^{\mathbf{f}} X$ , where roughly speaking  $\partial_+^{\mathbf{f}} X$  is the component of  $\partial X$  lying over  $Y^\circ$ , and  $\partial_-^{\mathbf{f}} X$  the component of  $\partial X$  lying over  $\partial Y$ . Then  $\mathbf{f}|_{\partial_+^{\mathbf{f}} X}$  is a strong submersion  $\mathbf{f}_+ : \partial_+^{\mathbf{f}} X \rightarrow Y$ , and  $\mathbf{f}|_{\partial_-^{\mathbf{f}} X} = \iota \circ \mathbf{f}_-$ , where  $\mathbf{f}_- : \partial_-^{\mathbf{f}} X \rightarrow \partial Y$  is a strong submersion, and  $\iota : \partial Y \rightarrow Y$  is the natural immersion. In (19) we must replace  $[\partial X_a, \mathbf{f}_a|_{\partial X_a}, \mathbf{C}_a|_{\partial X_a}]$  by  $[\partial_+^{\mathbf{f}_a} X_a, \mathbf{f}_{a,+}, \mathbf{C}_a|_{\partial_+^{\mathbf{f}_a} X_a}]$ .

In [12, §4.7] we define *cup* and *cap products*.

**Definition 4.3.** Let  $Y$  be an orbifold without boundary, and  $R$  a  $\mathbb{Q}$ -algebra. Define the *cup product*  $\cup : KC^k(Y; R) \times KC^l(Y; R) \rightarrow KC^{k+l}(Y; R)$  by

$$[X, \mathbf{f}, \mathbf{C}] \cup [\tilde{X}, \tilde{\mathbf{f}}, \tilde{\mathbf{C}}] = [X \times_{\mathbf{f}, Y, \tilde{\mathbf{f}}} \tilde{X}, \pi_Y, \mathbf{C} \times_Y \tilde{\mathbf{C}}],$$

extended  $R$ -bilinearly. Here  $\pi_Y : X \times_{\mathbf{f}, Y, \tilde{\mathbf{f}}} \tilde{X} \rightarrow Y$  is the projection from the fibre product, which is a strong submersion as both  $\mathbf{f}, \tilde{\mathbf{f}}$  are, and  $\mathbf{C} \times_Y \tilde{\mathbf{C}}$  is as in Theorem 4.1(g). Then  $\cup$  takes relations (i)–(iii) in both  $KC^k(Y; R)$  and  $KC^l(Y; R)$  to the same relations in  $KC^{k+l}(Y; R)$ . Thus  $\cup$  is well-defined.

Theorem 4.1(g) and (13)–(15) imply that for  $\gamma \in KC^k(Y; R)$ ,  $\delta \in KC^l(Y; R)$  and  $\epsilon \in KC^m(Y; R)$  we have

$$\gamma \cup \delta = (-1)^{kl} \delta \cup \gamma, \quad (21)$$

$$d(\gamma \cup \delta) = (d\gamma) \cup \delta + (-1)^k \gamma \cup (d\delta) \quad \text{and} \quad (\gamma \cup \delta) \cup \epsilon = \gamma \cup (\delta \cup \epsilon). \quad (22)$$

Therefore in the usual way  $\cup$  induces an associative, supercommutative product  $\cup : KH^k(Y; R) \times KH^l(Y; R) \rightarrow KH^{k+l}(Y; R)$  given for  $\gamma \in KC^k(Y; R)$  and  $\delta \in KC^l(Y; R)$  with  $d\gamma = d\delta = 0$  by

$$(\gamma + \text{Im } d_{k-1}) \cup (\delta + \text{Im } d_{l-1}) = (\gamma \cup \delta) + \text{Im } d_{k+l-1}.$$

Now suppose  $Y$  is compact. Then  $\text{id}_Y : Y \rightarrow Y$  is a (strong) submersion, with a trivial coorientation giving the positive sign to the zero vector bundle over  $Y$ . One can define natural co-gauge-fixing data  $\mathbf{C}_Y$  for  $(Y, \text{id}_Y)$  such that  $[Y, \text{id}_Y, \mathbf{C}_Y] \in KC^0(Y; R)$ , with  $d[Y, \text{id}_Y, \mathbf{C}_Y] = 0$ , and for all  $[X, \mathbf{f}, \mathbf{C}] \in KC^k(Y; R)$  we have

$$[Y, \text{id}_Y, \mathbf{C}_Y] \cup [X, \mathbf{f}, \mathbf{C}] = [X, \mathbf{f}, \mathbf{C}] \cup [Y, \text{id}_Y, \mathbf{C}_Y] = [X, \mathbf{f}, \mathbf{C}].$$

Thus  $[Y, \text{id}_Y, \mathbf{C}_Y]$  is the identity for  $\cup$ , at the cochain level. Passing to cohomology,  $[[Y, \text{id}_Y, \mathbf{C}_Y]] \in KH^0(Y; R)$  is the identity for  $\cup$  in  $KH^*(Y; R)$ . We call  $[[Y, \text{id}_Y, \mathbf{C}_Y]]$  the *fundamental class* of  $Y$ .

Define the *cap product*  $\cap : KC_k(Y; R) \times KC^l(Y; R) \rightarrow KC_{k-l}(Y; R)$  by

$$[X, \mathbf{f}, \mathbf{G}] \cap [\tilde{X}, \tilde{\mathbf{f}}, \tilde{\mathbf{C}}] = [X \times_{\mathbf{f}, Y, \tilde{\mathbf{f}}} \tilde{X}, \pi_Y, \mathbf{G} \times_Y \tilde{\mathbf{C}}],$$

extended  $R$ -bilinearly, with  $\mathbf{G} \times_Y \tilde{\mathbf{C}}$  as in Theorem 4.1(h). For  $\gamma \in KC_k(Y; R)$  and  $\delta, \epsilon \in KC^*(Y; R)$ , the analogue of (22), using Theorem 4.1(h) and (16), is

$$\partial(\gamma \cap \delta) = (\partial\gamma) \cap \delta + (-1)^{\dim Y - k} \gamma \cap (d\delta), \quad (\gamma \cap \delta) \cap \epsilon = \gamma \cap (\delta \cup \epsilon).$$

If also  $Y$  is compact then  $\gamma \cap [Y, \text{id}_Y, \mathbf{C}_Y] = \gamma$ . Thus  $\cap$  induces a *cap product*  $\cap : KH_k(Y; R) \times KH^l(Y; R) \rightarrow KH_{k-l}(Y; R)$ . These products  $\cap$  make Kuranishi chains and homology into *modules* over Kuranishi cochains and cohomology.

Let  $Y, Z$  be orbifolds without boundary, and  $h : Y \rightarrow Z$  a smooth, proper map. Then [12, Prop. 3.33] implies that pullbacks  $h^*$  and pushforwards  $h_*$  are compatible with  $\cup, \cap$  on (co)chains, in the sense that if  $\alpha \in KC_*(Y; R)$  and  $\beta, \gamma \in KC^*(Z; R)$  then

$$h^*(\beta \cup \gamma) = h^*(\beta) \cup h^*(\gamma) \quad \text{and} \quad h_*(\alpha \cap h^*(\beta)) = h_*(\alpha) \cap \beta. \quad (23)$$

Since  $\cup, \cap, h^*, h_*$  are compatible with  $d, \partial$ , passing to (co)homology shows that (23) also holds for  $\alpha \in KH_*(Y; R)$  and  $\beta, \gamma \in KH^*(Z; R)$ . If  $Z$  is compact then  $Y$  is, with  $h^*([Z, \text{id}_Z, \mathbf{C}_Z]) = [Y, \text{id}_Y, \mathbf{C}_Y]$  in  $KC^0(Y; R)$  and  $h^*([Z, \text{id}_Z, \mathbf{C}_Z]) = [[Y, \text{id}_Y, \mathbf{C}_Y]]$  in  $KH^0(Y; R)$ .

To summarize: Kuranishi cochains  $KC^*(Y; R)$  form a *supercommutative, associative, differential graded  $R$ -algebra*, and Kuranishi cohomology  $KH^*(Y; R)$  is a *supercommutative, associative, graded  $R$ -algebra*. These algebras are *with identity* if  $Y$  is compact without boundary, and *without identity* otherwise. Pullbacks  $h^*$  induce *algebra morphisms* on both cochains and cohomology. Kuranishi chains  $KC_*(Y; R)$  are a *graded module* over  $KC^*(Y; R)$ , and Kuranishi homology  $KH_*(Y; R)$  is a *graded module* over  $KH^*(Y; R)$ .

### 4.3 Poincaré duality, and isomorphism with $H_{\text{cs}}^*(Y; R)$

Suppose  $Y$  is an oriented manifold, of dimension  $n$ , without boundary, and not necessarily compact, and  $R$  is a commutative ring. Then as in Bredon [4, §VI.9] there are *Poincaré duality isomorphisms*

$$\text{Pd} : H_{\text{cs}}^k(Y; R) \longrightarrow H_{n-k}^{\text{si}}(Y; R) \quad (24)$$

between compactly-supported cohomology, and singular homology. If  $Y$  is also *compact* then it has a *fundamental class*  $[Y] \in H_n(Y; R)$ , and we can write the Poincaré duality map  $\text{Pd}$  of (24) in terms of the cap product by  $\text{Pd}(\alpha) = [Y] \cap \alpha$  for  $\alpha \in H_{\text{cs}}^k(Y; R)$ . Satake [18, Th. 3] showed that Poincaré duality isomorphisms (24) exist when  $Y$  is an oriented orbifold without boundary and  $R$  is a  $\mathbb{Q}$ -algebra.

Let  $Y$  be an orbifold of dimension  $n$  without boundary, and  $R$  a  $\mathbb{Q}$ -algebra. We wish to construct an isomorphism  $\Pi_{\text{cs}}^{\text{Kch}} : H_{\text{cs}}^*(Y; R) \rightarrow KH^*(Y; R)$  from compactly-supported cohomology to Kuranishi cohomology. In the case in which  $Y$  is oriented we will define  $\Pi_{\text{cs}}^{\text{Kch}}$  to be the composition

$$H_{\text{cs}}^k(Y; R) \xrightarrow{\text{Pd}} H_{n-k}^{\text{si}}(Y; R) \xrightarrow{\Pi_{\text{si}}^{\text{Kch}}} KH_{n-k}(Y; R) \xrightarrow{\Pi_{\text{Kh}}^{\text{Kch}}} KH^k(Y; R), \quad (25)$$

where the isomorphism  $\text{Pd}$  is as in (24), and  $\Pi_{\text{si}}^{\text{Kch}}$  is as in (18) and is an isomorphism by Theorem 3.3, and  $\Pi_{\text{Kh}}^{\text{Kch}}$  is an isomorphism between Kuranishi (co)homology, with inverse  $\Pi_{\text{Kh}}^{\text{Kch}} : KH^k(Y; R) \rightarrow KH_{n-k}(Y; R)$ .

These  $\Pi_{\text{Kch}}^{\text{Kh}}, \Pi_{\text{Kch}}^{\text{Kch}}$  are defined in [12, Def. 4.14]. At the (co)chain level, we define  $\Pi_{\text{Kch}}^{\text{Kch}} : KC^k(Y; R) \rightarrow KC_{n-k}(Y; R)$  by  $\Pi_{\text{Kch}}^{\text{Kch}} : [X, \mathbf{f}, \mathbf{C}] \mapsto [X, \mathbf{f}, \mathbf{G}_{\mathbf{C}}]$ , where  $\mathbf{G}_{\mathbf{C}}$  is gauge-fixing data for  $(X, \mathbf{f})$  constructed from the co-gauge-fixing data  $\mathbf{C}$  is a functorial way, and as  $\mathbf{f} : X \rightarrow Y$  is cooriented and  $Y$  is oriented, we obtain an orientation for  $X$  as in §2.7. Then  $\partial \circ \Pi_{\text{Kch}}^{\text{Kch}} = \Pi_{\text{Kch}}^{\text{Kh}} \circ \text{d}$ , so they induce morphisms  $\Pi_{\text{Kch}}^{\text{Kch}} : KH^k(Y; R) \rightarrow KH_{n-k}(Y; R)$  in (co)homology.

For  $\Pi_{\text{Kh}}^{\text{Kch}}$  the story is more complicated. To define  $\Pi_{\text{Kh}}^{\text{Kch}} : KC_{n-k}(Y; R) \rightarrow KC^k(Y; R)$  we cannot simply map  $[X, \mathbf{f}, \mathbf{G}] \mapsto [X, \mathbf{f}, \mathbf{C}_{\mathbf{G}}]$  for some co-gauge-fixing data  $\mathbf{C}_{\mathbf{G}}$  constructed from  $\mathbf{G}$ , since  $\mathbf{f}$  need only be strongly smooth for  $[X, \mathbf{f}, \mathbf{G}] \in KC_{n-k}(Y; R)$ , but  $\mathbf{f}$  must be a strong submersion for  $[X, \mathbf{f}, \mathbf{C}_{\mathbf{G}}] \in KC^k(Y; R)$ . Instead, we define  $\Pi_{\text{Kh}}^{\text{Kch}} : KC_{n-k}(Y; R) \rightarrow KC^k(Y; R)$  by  $\Pi_{\text{Kh}}^{\text{Kch}} : [X, \mathbf{f}, \mathbf{G}] \mapsto [X^Y, \mathbf{f}^Y, \mathbf{C}_{\mathbf{G}}^Y]$ . Here  $X^Y$  is  $X$  equipped with an *alternative Kuranishi structure*, which roughly speaking adds copies of  $\mathbf{f}^*(TY)$  to both the tangent bundle and obstruction bundle of  $X$ . Also  $\mathbf{f}^Y : X^Y \rightarrow Y$  is a lift of  $\mathbf{f}$  to  $X^Y$ , which is a strong submersion, and  $\mathbf{C}_{\mathbf{G}}^Y$  is co-gauge-fixing data for  $(X^Y, \mathbf{f}^Y)$  constructed in a functorial way from  $\mathbf{G}$ . Then  $\text{d} \circ \Pi_{\text{Kh}}^{\text{Kch}} = \Pi_{\text{Kh}}^{\text{Kch}} \circ \partial$ , so they induce morphisms  $\Pi_{\text{Kh}}^{\text{Kch}} : KH_{n-k}(Y; R) \rightarrow KH^k(Y; R)$  on (co)homology.

In [12, Th. 4.15] we show that these morphisms  $\Pi_{\text{Kch}}^{\text{Kh}}, \Pi_{\text{Kch}}^{\text{Kch}}$  on Kuranishi (co)homology are inverse. Thus the third morphism  $\Pi_{\text{Kh}}^{\text{Kch}}$  in (25) is an isomorphism, so the composition  $\Pi_{\text{cs}}^{\text{Kch}} : H_{\text{cs}}^k(Y; R) \rightarrow KH^k(Y; R)$  is an isomorphism. Changing the orientation of  $Y$  changes the sign of  $\text{Pd}, \Pi_{\text{Kch}}^{\text{Kch}}$ , and so does not change  $\Pi_{\text{cs}}^{\text{Kch}}$ . If  $Y$  is not orientable we can make a similar argument using homology groups  $H_{n-k}^{\text{si}}(Y; O \times_{\{\pm 1\}} R)$ ,  $KH_{n-k}(Y; O \times_{\{\pm 1\}} R)$  twisted by the principal  $\mathbb{Z}_2$ -bundle  $O$  of orientations on  $Y$ . Thus we prove [12, Cor. 4.17]:

**Theorem 4.4.** *Let  $Y$  be an orbifold without boundary, and  $R$  a  $\mathbb{Q}$ -algebra. Then there are natural isomorphisms  $\Pi_{\text{cs}}^{\text{Kch}} : H_{\text{cs}}^k(Y; R) \rightarrow KH^k(Y; R)$  for  $k \geq 0$ , and  $KH^k(Y; R) = 0$  when  $k < 0$ .*

In [12, §4.5] the theorem is extended to  $Y$  with boundary, going via relative homology  $H_*^{\text{si}}(Y, \partial Y; R), KH_*(Y, \partial Y; R)$ . In [12, Th. 4.34] we show that

the isomorphisms  $\Pi_{\text{cs}}^{\text{Kch}} : H_{\text{cs}}^*(Y; R) \rightarrow KH^*(Y; R)$  and  $\Pi_{\text{si}}^{\text{Kch}} : H_{\text{si}}^*(Y; R) \rightarrow KH_*(Y; R)$  in Theorems 3.3 and 4.4 identify the cup and cap products  $\cup, \cap$  on  $H_{\text{cs}}^*(Y; R), H_{\text{si}}^*(Y; R)$  with those on  $KH^*(Y; R), KH_*(Y; R)$ .

## 5 Kuranishi bordism and cobordism

We now summarize parts of [12, §5] on Kuranishi (co)bordism. They are based on the classical bordism theory introduced by Atiyah [3]. In fact [12, §5] studies five different kinds of Kuranishi (co)bordism, but we discuss only one.

### 5.1 Classical bordism and cobordism groups

Bordism groups were introduced by Atiyah [3], and Connor [6, §I] gives a good introduction. Our definition is not standard, but fits in with §5.2.

**Definition 5.1.** Let  $Y$  be an orbifold without boundary. Consider pairs  $(X, f)$ , where  $X$  is a compact, oriented manifold without boundary or corners, not necessarily connected, and  $f : X \rightarrow Y$  is a smooth map. An *isomorphism* between two such pairs  $(X, f), (\tilde{X}, \tilde{f})$  is an orientation-preserving diffeomorphism  $i : X \rightarrow \tilde{X}$  with  $f = \tilde{f} \circ i$ . Write  $[X, f]$  for the isomorphism class of  $(X, f)$ .

Let  $R$  be a commutative ring. For each  $k \geq 0$ , define the  $k^{\text{th}}$  *bordism group*  $B_k(Y; R)$  of  $Y$  with coefficients in  $R$  to be the  $R$ -module of finite  $R$ -linear combinations of isomorphism classes  $[X, f]$  for which  $\dim X = k$ , with the relations:

- (i)  $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$  for all classes  $[X, f], [X', f']$ ; and
- (ii) Suppose  $Z$  is a compact, oriented  $(k+1)$ -manifold with boundary but without (g-)corners, and  $g : Z \rightarrow Y$  is smooth. Then  $[\partial Z, g|_{\partial Z}] = 0$ .

Here is how this definition relates to those in [3, 6]. When  $Y$  is a manifold and  $R = \mathbb{Z}$ , our  $B_k(Y; \mathbb{Z})$  is equivalent to Connor's *differential bordism group*  $D_k(Y)$ , [6, §I.9]. Atiyah [3, §2] and Connor [6, §I.4] also define *bordism groups*  $MSO_k(Y)$  as for  $B_k(Y; \mathbb{Z})$  above, but only requiring  $f : X \rightarrow Y$  to be continuous, not smooth. Connor [6, Th. I.9.1] shows that when  $Y$  is a manifold, the natural projection  $D_k(Y) \rightarrow MSO_k(Y)$  is an isomorphism.

As in [6, §I.5], bordism is a *generalized homology theory*, that is, it satisfies all the Eilenberg–Steenrod axioms for a homology theory except the dimension axiom. The bordism groups of a point  $MSO_*(\text{pt})$  are known, [6, §I.2]. This gives some information on bordism groups of general spaces  $Y$ : for any generalized homology theory  $GH_*(Y)$ , there is a spectral sequence from the singular homology  $H_{\text{si}}^*(Y; GH_*(\text{pt}))$  of  $Y$  with coefficients in  $GH_*(\text{pt})$  converging to  $GH_*(Y)$ , so that  $GH_*(\mathcal{S}^n) \cong H_{\text{si}}^*(\mathcal{S}^n; GH_*(\text{pt}))$ , for instance.

Atiyah [3] and Connor [6, §13] also define *cobordism groups*  $MSO^k(Y)$  for  $k \in \mathbb{Z}$ , which are a *generalized cohomology theory* dual to bordism  $MSO_k(Y)$ . There is a natural product  $\cup$  on  $MSO^*(Y)$ , making it into a supercommutative

ring. If  $Y$  is a compact, oriented  $n$ -manifold without boundary then [3, Th. 3.6], [6, Th. 13.4] there are canonical Poincaré duality isomorphisms

$$MSO^k(Y) \cong MSO_{n-k}(Y) \quad \text{for } k \in \mathbb{Z}. \quad (26)$$

The definition of  $MSO^*(Y)$  uses homotopy theory, direct limits of  $k$ -fold suspensions, and classifying spaces. There does not seem to be a satisfactory differential-geometric definition of cobordism groups parallel to Definition 5.1.

## 5.2 Kuranishi bordism and cobordism groups

Motivated by §5.1, following [12, §5.2] we define:

**Definition 5.2.** Let  $Y$  be an orbifold. Consider pairs  $(X, \mathbf{f})$ , where  $X$  is a compact, oriented Kuranishi space without boundary, and  $\mathbf{f} : X \rightarrow Y$  is strongly smooth. An *isomorphism* between two pairs  $(X, \mathbf{f}), (\tilde{X}, \tilde{\mathbf{f}})$  is an orientation-preserving strong diffeomorphism  $\mathbf{i} : X \rightarrow \tilde{X}$  with  $\tilde{\mathbf{f}} = \mathbf{f} \circ \mathbf{i}$ . Write  $[X, \mathbf{f}]$  for the isomorphism class of  $(X, \mathbf{f})$ .

Let  $R$  be a commutative ring. For each  $k \in \mathbb{Z}$ , define the  $k^{\text{th}}$  *Kuranishi bordism group*  $KB_k(Y; R)$  of  $Y$  with coefficients in  $R$  to be the  $R$ -module of finite  $R$ -linear combinations of isomorphism classes  $[X, \mathbf{f}]$  for which  $\text{vdim } X = k$ , with the relations:

- (i)  $[X, \mathbf{f}] + [X', \mathbf{f}'] = [X \amalg X', \mathbf{f} \amalg \mathbf{f}']$  for all classes  $[X, \mathbf{f}], [X', \mathbf{f}']$ ; and
- (ii) Suppose  $W$  is a compact, oriented Kuranishi space with boundary but without (g-)corners, with  $\text{vdim } W = k + 1$ , and  $\mathbf{e} : W \rightarrow Y$  is strongly smooth. Then  $[\partial W, \mathbf{e}|_{\partial W}] = 0$ .

Elements of  $KB_k(Y; R)$  will be called *Kuranishi bordism classes*.

Let  $h : Y \rightarrow Z$  be a smooth map of orbifolds. Define the *pushforward*  $h_* : KB_k(Y; R) \rightarrow KB_k(Z; R)$  by  $h_* : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto \sum_{a \in A} \rho_a [X_a, h \circ \mathbf{f}_a]$ . This takes relations (i), (ii) in  $KB_k(Y; R)$  to (i), (ii) in  $KB_k(Z; R)$ , and so is well-defined. Pushforward is functorial, that is,  $(g \circ h)_* = g_* \circ h_*$ .

Now Kuranishi bordism  $KB_*(Y; R)$  is like Kuranishi homology  $KH_*(Y; R)$  in §3.2, but using Kuranishi spaces  $X$  without boundary, and omitting gauge-fixing data  $\mathbf{G}$ . Thus it seems natural to define Kuranishi cobordism  $KB^*(Y; R)$  by modifying the definition of Kuranishi cohomology  $KH^*(Y; R)$  in §4.2 in the same way, following [12, §5.4–§5.5].

**Definition 5.3.** Let  $Y$  be an orbifold without boundary. Consider pairs  $(X, \mathbf{f})$ , where  $X$  is a compact Kuranishi space without boundary, and  $\mathbf{f} : X \rightarrow Y$  is a cooriented strong submersion. An *isomorphism* between two pairs  $(X, \mathbf{f}), (\tilde{X}, \tilde{\mathbf{f}})$  is a coorientation-preserving strong diffeomorphism  $\mathbf{i} : X \rightarrow \tilde{X}$  with  $\tilde{\mathbf{f}} = \mathbf{f} \circ \mathbf{i}$ . Write  $[X, \mathbf{f}]$  for the isomorphism class of  $(X, \mathbf{f})$ .

Let  $R$  be a commutative ring. For each  $k \in \mathbb{Z}$ , define the  $k^{\text{th}}$  *Kuranishi cobordism group*  $KB^k(Y; R)$  of  $Y$  with coefficients in  $R$  to be the  $R$ -module of finite  $R$ -linear combinations of isomorphism classes  $[X, \mathbf{f}]$  for which  $\text{vdim } X = \dim Y - k$ , with the relations:



- (i)  $[X, \mathbf{f}] + [X', \mathbf{f}'] = [X \amalg X', \mathbf{f} \amalg \mathbf{f}']$  for all classes  $[X, \mathbf{f}], [X', \mathbf{f}']$ ; and
- (ii) Suppose  $W$  is a compact Kuranishi space with boundary but without (g-) corners, with  $\text{vdim } W = \dim Y - k + 1$ , and  $e : W \rightarrow Y$  is a cooriented strong submersion. Then  $e|_{\partial W} : \partial W \rightarrow Y$  is a cooriented strong submersion, and we impose the relation  $[\partial W, e|_{\partial W}] = 0$  in  $KB^k(Y; R)$ .

Elements of  $KB^k(Y; R)$  will be called *Kuranishi cobordism classes*.

Define the *cup product*  $\cup : KB^k(Y; R) \times KB^l(Y; R) \rightarrow KB^{k+l}(Y; R)$  by

$$\left[ \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \right] \cup \left[ \sum_{b \in B} \sigma_b [\tilde{X}_b, \tilde{\mathbf{f}}_b] \right] = \sum_{a \in A, b \in B} \rho_a \sigma_b [X_a \times_{\mathbf{f}_a, Y, \tilde{\mathbf{f}}_b} \tilde{X}_b, \pi_Y], \quad (27)$$

for  $A, B$  finite and  $\rho_a, \sigma_b \in R$ . The coorientations on  $(X_a, \mathbf{f}_a)$  and  $(\tilde{X}_b, \tilde{\mathbf{f}}_b)$  induce a coorientation on  $(X_a \times_Y \tilde{X}_b, \pi_Y)$  as in §2.7. Similarly, define the *cap product*  $\cap : KB_k(Y; R) \times KB^l(Y; R) \rightarrow KB_{k-l}(Y; R)$  by the same formula (27), where now  $[X_a, \mathbf{f}_a] \in KB_k(Y; R)$  so that  $\mathbf{f}_a$  is strongly smooth and  $X_a$  oriented, and the orientation on  $X_a$  and coorientation for  $\mathbf{f}_b$  combine to give an orientation for  $X_a \times_Y \tilde{X}_b$ .

One can show that  $\cup, \cap$  are well-defined, that  $\cup$  is associative and supercommutative, and that  $(\gamma \cap \delta) \cap \epsilon = \gamma \cap (\delta \cup \epsilon)$  for  $\gamma \in KB_*(Y; R)$  and  $\delta, \epsilon \in KB^*(Y; R)$ . If  $Y$  is also compact then using the trivial coorientation for  $\text{id}_Y : Y \rightarrow Y$ , we have  $[Y, \text{id}_Y] \in KB^0(Y; R)$ , which is the *identity* for  $\cup$  and  $\cap$ . Thus,  $KB^*(Y; R)$  is a *graded, supercommutative, associative  $R$ -algebra, with identity* if  $Y$  is compact, and *without identity* otherwise, and  $\cap$  makes  $KB_*(Y; R)$  into a module over  $KB^*(Y; R)$ .

Let  $Y, Z$  be orbifolds without boundary, and  $h : Y \rightarrow Z$  a smooth, proper map. Motivated by (20), define the *pullback*  $h^* : KB^k(Z; R) \rightarrow KB^k(Y; R)$  by  $h^* : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto \sum_{a \in A} \rho_a [Y \times_{h, Z, \mathbf{f}_a} X_a, \pi_Y]$ . This takes relations (i), (ii) in  $KB^k(Z; R)$  to (i), (ii) in  $KB^k(Y; R)$ , and so is well-defined. Pullbacks are functorial,  $(g \circ h)^* = h^* \circ g^*$ . The cup and cap products are compatible with pullbacks and pushforwards, as in (23).

### 5.3 Morphisms to and from Kuranishi (co)bordism

In [12, §5.3–§5.4] we define morphisms between these groups.

**Definition 5.4.** Let  $Y$  be an orbifold, and  $R$  a commutative ring. Define morphisms  $\Pi_{\text{bo}}^{\text{Kb}} : B_k(Y; R) \rightarrow KB_k(Y; R)$  for  $k \geq 0$  by  $\Pi_{\text{bo}}^{\text{Kb}} : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a]$ , interpreting the manifold  $X_a$  as a Kuranishi space, and the smooth map  $\mathbf{f}_a : X_a \rightarrow Y$  as strongly smooth.

Define morphisms  $\Pi_{\text{Kb}}^{\text{Kh}} : KB_k(Y; R) \rightarrow KH_k(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$  for  $k \in \mathbb{Z}$  by  $\Pi_{\text{Kb}}^{\text{Kh}} : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto [\sum_{a \in A} \pi(\rho_a) [X_a, \mathbf{f}_a, \mathbf{G}_a]]$ , where  $\mathbf{G}_a$  is some choice of gauge-fixing data for  $(X_a, \mathbf{f}_a)$ , which exists by Theorem 3.1(a), and  $\pi : R \rightarrow R \otimes_{\mathbb{Z}} \mathbb{Q}$  is the natural morphism. Using Theorem 3.1(e) over  $[0, 1] \times X_a$  one can show that  $\Pi_{\text{Kb}}^{\text{Kh}}$  is independent of the choice of  $\mathbf{G}_a$ , and is well-defined.

Similarly, define morphisms  $\Pi_{\text{Kcb}}^{\text{Kch}} : KB^k(Y; R) \rightarrow KH^k(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$  for  $k \in \mathbb{Z}$  by  $\Pi_{\text{Kcb}}^{\text{Kch}} : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto [\sum_{a \in A} \pi(\rho_a) [X_a, \mathbf{f}_a, \mathbf{C}_a]]$ , where  $\mathbf{C}_a$  is some choice of co-gauge-fixing data for  $(X_a, \mathbf{f}_a)$ .

These  $\Pi_{\text{Kcb}}^{\text{Kch}}, \Pi_{\text{Kcb}}^{\text{Kh}}$  take cup and cap products  $\cup, \cap$  on  $KB^*, KB_*(Y; R)$  to  $\cup, \cap$  on  $KH^*, KH_*(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$ , and if  $Y$  is compact, they take the identity  $[Y, \text{id}_Y] \in KB^0(Y; R)$  to the identity  $[Y, \text{id}_Y, \mathbf{C}_Y] \in KH^0(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$ .

Consider the sequence of morphisms

$$B_k(Y; R) \xrightarrow{\Pi_{\text{bo}}^{\text{Kb}}} KB_k(Y; R) \xrightarrow{\Pi_{\text{Kcb}}^{\text{Kh}}} KH_k(Y; R \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{(\Pi_{\text{si}}^{\text{Kh}})^{-1}} H_k^{\text{si}}(Y; R \otimes_{\mathbb{Z}} \mathbb{Q}),$$

where  $(\Pi_{\text{si}}^{\text{Kh}})^{-1}$  exists by Theorem 3.3. The composition is the natural map  $B_k(Y; R) \rightarrow H_k^{\text{si}}(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$  taking  $[X, f] \mapsto f_*([X])$ . Thus we find:

**Corollary 5.5.** *Let  $Y$  be an orbifold, and  $R$  a commutative ring. Then  $KB_k(Y; R)$  is at least as large as the image of  $B_k(Y; R)$  in  $H_k^{\text{si}}(Y; R \otimes_{\mathbb{Z}} \mathbb{Q})$ .*

We will see in §5.4 that  $KB_*(Y; R)$  is actually very large.

The Poincaré duality story for Kuranishi (co)homology in §4.3 has an analogue for Kuranishi (co)bordism, as in [12, §5.4]. Let  $Y$  be an oriented  $n$ -orbifold without boundary, and  $R$  a commutative ring. Define  $R$ -module morphisms  $\Pi_{\text{Kcb}}^{\text{Kb}} : KB^k(Y; R) \rightarrow KB_{n-k}(Y; R)$  for  $k \in \mathbb{Z}$  by  $\Pi_{\text{Kcb}}^{\text{Kb}} : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a]$ , using the coorientation for  $\mathbf{f}_a$  from  $[X_a, \mathbf{f}_a] \in KB^k(Y; R)$  and the orientation on  $Y$  to determine the orientation on  $X_a$  for  $[X_a, \mathbf{f}_a] \in KB_{n-k}(Y; R)$ .

Define  $\Pi_{\text{Kcb}}^{\text{Kcb}} : KB_{n-k}(Y; R) \rightarrow KB^k(Y; R)$  by  $\Pi_{\text{Kcb}}^{\text{Kcb}} : \sum_{a \in A} \rho_a [X_a, \mathbf{f}_a] \mapsto \sum_{a \in A} \rho_a [X_a^Y, \mathbf{f}_a^Y]$ , where  $X_a^Y$  is  $X_a$  with an *alternative Kuranishi structure* as in §4.3, and  $\mathbf{f}_a^Y : X_a^Y \rightarrow Y$  is a lift of  $\mathbf{f}_a$  to  $X_a^Y$ , which is a strong submersion. Then [12, Th. 5.11] shows that  $\Pi_{\text{Kcb}}^{\text{Kcb}}$  and  $\Pi_{\text{Kcb}}^{\text{Kb}}$  are inverses, so they are both isomorphisms. Using this and ideas in §5.1 including (26), if  $Y$  is a compact manifold we can define a natural morphism  $\Pi_{\text{cb}}^{\text{Kcb}} : MSO^*(Y) \rightarrow KB^*(Y; \mathbb{Z})$ , so Kuranishi cobordism is a generalization of classical cobordism.

## 5.4 How large are Kuranishi (co)bordism groups?

Theorems 3.3 and 4.4 showed that Kuranishi (co)homology are isomorphic to classical (compactly-supported) (co)homology, so they are not new topological invariants. In contrast, Kuranishi (co)bordism are not isomorphic to classical (co)bordism, they are genuinely new topological invariants, so it is interesting to ask what we can say about them. We now summarize the ideas of [12, §5.6–§5.7], which show that  $KB_*(Y; R)$  and  $KB^*(Y; R)$  are *very large* for any orbifold  $Y$  and commutative ring  $R$  with  $Y \neq \emptyset$  and  $R \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ .

One reason for this is that in a class  $\sum_{a \in A} \rho_a [X_a, \mathbf{f}_a]$  in  $KB_k(Y; R)$  there is a lot of information stored in the *orbifold strata* of  $X_a$  for  $a \in A$ . We define these for orbifolds, [12, Def. 5.15].

**Definition 5.6.** Let  $\Gamma$  be a finite group, and consider (finite-dimensional) real representations  $(W, \omega)$  of  $\Gamma$ , that is,  $W$  is a finite-dimensional real vector space

and  $\omega : \Gamma \rightarrow \text{Aut}(W)$  is a group morphism. Call  $(W, \omega)$  a *trivial representation* if  $\omega \equiv \text{id}_W$ , and a *nontrivial representation* if  $\text{Fix}(\omega(\Gamma)) = \{0\}$ . Then every  $\Gamma$ -representation  $(W, \omega)$  has a unique decomposition  $W = W^t \oplus W^{\text{nt}}$  as the direct sum of a trivial representation  $(W^t, \omega^t)$  and a nontrivial representation  $(W^{\text{nt}}, \omega^{\text{nt}})$ , where  $W^t = \text{Fix}(\omega(\Gamma))$ .

Now let  $X$  be an  $n$ -orbifold,  $\Gamma$  be a finite group, and  $\rho$  be an *isomorphism class of nontrivial  $\Gamma$ -representations*. Each  $p \in X$  has a *stabilizer group*  $\text{Stab}_X(p)$ . The tangent space  $T_p X$  is an  $n$ -dimensional vector space with a representation  $\tau_p$  of  $\text{Stab}_X(p)$ . Let  $\lambda : \Gamma \rightarrow \text{Stab}_X(p)$  be an injective group morphism, so that  $\lambda(\Gamma)$  is a subgroup of  $\text{Stab}_X(p)$  isomorphic to  $\Gamma$ . Hence  $\tau_p \circ \lambda : \Gamma \rightarrow \text{Aut}(T_p X)$  is a  $\Gamma$ -representation, and we can split  $T_p X = (T_p X)^t \oplus (T_p X)^{\text{nt}}$  into trivial and nontrivial  $\Gamma$ -representations, and form the isomorphism class  $[(T_p X)^{\text{nt}}, (\tau_p \circ \lambda)^{\text{nt}}]$ . As a set, define the *orbifold stratum*  $X^{\Gamma, \rho}$  to be

$$X^{\Gamma, \rho} = \{ \text{Stab}_X(p) \cdot (p, \lambda) : p \in X, \lambda : \Gamma \rightarrow \text{Stab}_X(p) \text{ is an injective group morphism, } [(T_p X)^{\text{nt}}, (\tau_p \circ \lambda)^{\text{nt}}] = \rho \},$$

where  $\text{Stab}_X(p)$  acts on pairs  $(p, \lambda)$  by  $\sigma : (p, \lambda) \mapsto (p, \lambda^\sigma)$ , where  $\lambda^\sigma : \Gamma \rightarrow \text{Stab}_X(p)$  is given by  $\lambda^\sigma(\gamma) = \sigma \lambda(\gamma) \sigma^{-1}$ . Define a map  $\iota^{\Gamma, \rho} : X^{\Gamma, \rho} \rightarrow X$  by  $\iota^{\Gamma, \rho} : \text{Stab}_X(p) \cdot (p, \lambda) \mapsto p$ . Then [12, Prop. 5.16] shows that  $X^{\Gamma, \rho}$  is an orbifold of dimension  $n - \dim \rho$ , and  $\iota^{\Gamma, \rho}$  lifts to a proper, finite immersion.

If  $X$  is a Kuranishi space, there is a parallel definition [12, Def. 5.18] of orbifold strata  $X^{\Gamma, \rho}$ , which we will not give. The most important difference is that  $\rho$  is now a *virtual nontrivial representation* of  $\Gamma$ , that is, a formal difference of nontrivial representations, so that  $\dim \rho \in \mathbb{Z}$  rather than  $\dim \rho \in \mathbb{N}$ . We find [12, Prop. 5.19] that  $X^{\Gamma, \rho}$  is a Kuranishi space with  $\text{vdim } X^{\Gamma, \rho} = \text{vdim } X - \dim \rho$ , equipped with a proper, finite, strongly smooth map  $\iota^{\Gamma, \rho} : X^{\Gamma, \rho} \rightarrow X$ .

We would like to define projections  $\Pi^{\Gamma, \rho} : KB_k(Y; R) \rightarrow KB_{k - \dim \rho}(Y; R)$  mapping  $\Pi^{\Gamma, \rho} : [X_a, \mathbf{f}_a] \rightarrow [X_a^{\Gamma, \rho}, \mathbf{f}_a|_{X_a^{\Gamma, \rho}}]$ . But there is a problem: we need to define an *orientation* on  $X_a^{\Gamma, \rho}$  from the orientation on  $X_a$ , and for general  $\Gamma, \rho$  this may not be possible. To overcome this we suppose  $|\Gamma|$  is odd, which implies that  $\dim \rho$  is even for all  $\rho$ , and there is then a consistent way to define orientations on  $X_a^{\Gamma, \rho}$ , and  $\Pi^{\Gamma, \rho}$  is well-defined.

Let  $Y$  be a nonempty, connected orbifold. In [12, §5.7], for each finite group  $\Gamma$  with  $|\Gamma|$  odd and all isomorphism classes  $\rho$  of virtual nontrivial representations of  $\Gamma$ , we construct a class  $C^{\Gamma, \rho} \in KB_{\dim \rho}(Y; \mathbb{Z})$ , such that  $\Pi_{\text{Kh}}^{\text{Kh}} \circ \Pi^{\Gamma, \rho}(C^{\Gamma, \rho})$  is nonzero in  $KH_0(Y; \mathbb{Q}) \cong H_0^{\text{si}}(Y; \mathbb{Q}) \cong \mathbb{Q}$ , and  $\Pi^{\Delta, \sigma}(C^{\Gamma, \rho}) = 0$  if either  $|\Delta| \geq |\Gamma|$  and  $\Delta \not\cong \Gamma$ , or if  $\Delta = \Gamma$  and  $\rho \neq \sigma$ . It follows that taken over all isomorphism classes of pairs  $\Gamma, \rho$ , the classes  $C^{\Gamma, \rho} \in KB_*(Y; \mathbb{Z})$  are *linearly independent over  $\mathbb{Z}$* . Extending to an arbitrary commutative ring  $R$ , and using the Poincaré duality ideas of §5.3, we deduce:

**Theorem 5.7.** *Let  $Y$  be a nonempty orbifold, and  $R$  a commutative ring with  $R \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ . Then  $KB_{2k}(Y; R)$  is infinitely generated over  $R$  for all  $k \in \mathbb{Z}$ . If also  $Y$  is oriented of dimension  $n$  then  $KB^{n-2k}(Y; R)$  is infinitely generated over  $R$  for all  $k \in \mathbb{Z}$ .*

Theorem 5.7 supports the idea that Kuranishi bordism, (or better, *almost complex Kuranishi bordism*, as in [12, Ch. 5]) may be a useful tool for studying (closed) Gromov–Witten invariants. In [12, §6.2] we define new Gromov–Witten type invariants  $[\bar{\mathcal{M}}_{g,m}(M, J, \beta), \prod_i \mathbf{ev}_i]$  in Kuranishi bordism  $KB_*(M^m; \mathbb{Z})$ . Theorem 5.7 indicates that  $KB_*(M^m; \mathbb{Z})$  is very large, so that these new invariants *contain a lot of information*, and that much of this information has to do with the *orbifold strata* of the moduli spaces  $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$ .

Also, these new invariants are defined in groups  $KB_*(M^m; \mathbb{Z})$  over  $\mathbb{Z}$ , not  $\mathbb{Q}$ . When we project to Kuranishi homology or singular homology to get conventional Gromov–Witten invariants, we must work in homology over  $\mathbb{Q}$ . The reason we cannot work over  $\mathbb{Z}$  is because of rational contributions from the orbifold strata of  $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$ . Kuranishi bordism looks like a good framework for describing these contributions, and so for understanding the integrality properties of Gromov–Witten invariants, such as the Gopakumar–Vafa Integrality Conjecture for Gromov–Witten invariants of Calabi–Yau 3-folds. This is discussed in [12, §6.3], and the author hopes to take it further in [15].

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